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ON THE NEWTON–KANTOROVICH THEOREM

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The Newton–Kantorovich theorem enjoys a special status, as it is both a fundamental result in Numerical Analysis, e.g., for providing an iterative method for computing the zeros of polynomials or of systems of nonlinear equations, and a fundamental result in Nonlinear Functional Analysis, e.g., for establishing that a nonlinear equation in an infinite-dimensional function space has a solution. Yet its detailed proof in full generality is not easy to locate in the literature.

The purpose of this article, which is partly expository in nature, is to carefully revisit this theorem, by means of a two-tier approach.

First, we give a detailed, and essentially self-contained, account of the classical proof of this theorem, which essentially relies on careful estimates based on the integral form of the mean value theorem for functions of class $\mathcal{C}^1$ with values in a Banach space, and on the so-called majorant method. Our treatment also includes a careful discussion of the often overlooked uniqueness issue. An example of a nonlinear two-point boundary value problem is also given that illustrates the power of this theorem for establishing an existence theorem when other methods of nonlinear functional analysis cannot be used.

Second, we give a new version of this theorem, the assumptions of which involve only one constant instead of three constants in its classical version and the proof of which is substantially simpler as it altogether avoids the majorant method. For these reasons, this new version, which captures all the basic features of the classical version could be considered as a good alternative to the classical Newton–Kantorovich theorem.

Keywords: Newton's method; Newton–Kantorovich theorem; solution of nonlinear equations; two-point boundary value problems.

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1. Introduction

The iterative method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k \geq 0, \quad \text{with } x_0 \in \mathbb{R} \text{ given},$$

where f is a differentiable real-valued function defined over an open subset of $\mathbb{R}$ is the well-known *Newton's method*. This method is indeed due to Sir Isaac Newton (1642–1727), who introduced it in 1669 for computing zeros of polynomials.

Given a differentiable mapping $f : \Omega \subset X \to Y$, where X and Y are now *arbitrary normed vector spaces* and Ω is an open subset of X, this simple case is the basis for the following *extension of Newton's method* for finding, and approximating, the *zeros* of f in Ω, i.e. those points $a \in \Omega$ such that $f(a) = 0$: Given an arbitrary point $x_0 \in \Omega$, the sequence $(x_k)_{k=0}^{\infty}$ is defined by

$$x_{k+1} = x_k - (f'(x_k))^{-1} f(x_k), \quad k \geq 0,$$

where $f'(x_k)$ now denotes the *Fréchet derivative* of the mapping f at the point x_k. Of course, this iterative method makes sense only if *all the points* x_k, which are called the *Newton iterates* for the mapping f, *remain in* Ω, and only if *the derivatives* $f'(x_k) \in \mathcal{L}(X;Y)$ *are invertible for all* $k \geq 0$.

Newton's method is especially well suited for solving *systems of n nonlinear equations in n unknowns*, which correspond to mappings $\boldsymbol{f} = (f_i) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. In this case, one iteration of Newton's method consists in finding the solution $\boldsymbol{\delta x}_k \in \mathbb{R}^n$ to the *linear system* $\boldsymbol{f}'(\boldsymbol{x}_k)\boldsymbol{\delta x}_k = -\boldsymbol{f}(\boldsymbol{x}_k)$, where $\boldsymbol{f}'(\boldsymbol{x}_k)$ denotes the $n \times n$ matrix $(\partial_j f_i(\boldsymbol{x}_k))$, and then in letting $\boldsymbol{x}_{k+1} := \boldsymbol{x}_k + \boldsymbol{\delta x}_k$.

In 1948, Kantorovich [7] published a powerful theorem, since then called the *Newton–Kantorovich theorem*, which gives sufficient conditions guaranteeing that such a Newton method *converges to a zero of f in* Ω. In other words, this theorem not only establishes the *convergence of Newton's method*, but also establishes the *existence of a zero of f*. Equally remarkable is the nature of the *assumptions* of this theorem, which all bear on the values of the function f and its derivative f' at the initial guess x_0 and on the behavior of f' in a neighborhood of x_0; hence, all these assumptions are in principle entirely verifiable *a priori*.

The well-known *Banach fixed point theorem* provides in a sense the simplest way to show that a nonlinear equation (written as $f(x) = x$) has a solution and to approximate this solution by means of an iterative method. The Newton–Kantorovich theorem thus provides another, but somewhat less simple, way to likewise establish *the existence of a solution to a nonlinear equation* (now written as $f(x) = 0$), together with an *iterative method for approximating such a solution*. Its proof requires only a *modicum* of linear and nonlinear functional analysis, *viz.*, the notion of *complete space* (like that of Banach fixed point theorem) and, in addition, a basic result of differential calculus in normed vector spaces, *viz.*, the *mean value theorem for functions of class* $\mathcal{C}^1$ *with values in a Banach space* (see Sec. 2.4).

2. Functional Analytic Preliminaries

In order to render this article as self-contained as possible, this section gathers the functional analytic preliminaries that will be needed in the proof of the Newton–Kantorovich theorem. These notions are found in classical textbooks such as [5] or [9].

2.1. *Normed vector spaces*

Given a normed vector space X (all spaces considered in this article are over $\mathbb{R}$), a point $x_0 \in X$, and $r > 0$, the open, respectively, closed, ball centered at x_0 with radius r is denoted $B(x_0; r)$, respectively, $\overline{B(x_0; r)}$.

Given two normed vector spaces X and Y, the notation $\mathcal{L}(X; Y)$, or simply $\mathcal{L}(X)$ if $X = Y$, denotes the vector space formed by all continuous linear operators from X into Y.

Let X be a Banach space, let Y be a normed vector space, and let $A \in \mathcal{L}(X; Y)$ be one-to-one and onto, with a continuous inverse A^{-1}, i.e. $A^{-1} \in \mathcal{L}(Y; X)$. Then, any $B \in \mathcal{L}(X; Y)$ such that $\|A^{-1}(B - A)\|_{\mathcal{L}(X)} < 1$ is also one-to-one and onto, with a continuous inverse $B^{-1} \in \mathcal{L}(Y; X)$ that satisfies

$$\|B^{-1}\|_{\mathcal{L}(Y;X)} \leq \frac{\|A^{-1}\|_{\mathcal{L}(Y;X)}}{1 - \|A^{-1}(B - A)\|_{\mathcal{L}(X)}}.$$

2.2. *The Fréchet derivative*

Let X and Y be normed vector spaces, and let Ω be an open subset of the space X. A mapping $f : \Omega \subset X \to Y$ is differentiable at a point $a \in \Omega$ if there exists an element $f'(a) \in \mathcal{L}(X; Y)$ such that

$$f(a + h) = f(a) + f'(a)h + \|h\|_X \delta(h) \quad \text{with } \lim_{h \to 0} \delta(h) = 0 \text{ in } Y.$$

The linear mapping $f'(a) \in \mathcal{L}(X; Y)$ defined in this fashion is unique and is called the *Fréchet derivative*, or simply the *derivative*, of the mapping f *at the point* a. If a mapping $f : \Omega \subset X \to Y$ is differentiable at all points of the open set Ω, it is said to be *differentiable in* Ω. If the mapping $f' : x \in \Omega \subset X \to f'(x) \in \mathcal{L}(X; Y)$, which is well defined in this case, is continuous, the mapping f is said to be *continuously differentiable in* Ω, or simply *of class* $\mathcal{C}^1$ *in* Ω. The space of all continuously differentiable mappings from Ω into Y is denoted $\mathcal{C}^1(\Omega; Y)$, or simply $\mathcal{C}^1(\Omega)$ if $Y = \mathbb{R}$.

2.3. *Definition of the integral* $\int_a^b g(\xi)\, \mathrm{d}\xi$, *where* g *is a continuous function* $g : [a, b] \subset \mathbb{R} \to Y$ *and* Y *is a Banach space*

This definition is carried out in two stages:

First, let $g : [a, b] \subset \mathbb{R} \to Y$ be a *step function over* $[a, b]$: This means that there exist finitely many points $\xi_i \in [a, b]$, $0 \leq i \leq n$, and vectors $c_i \in Y$, $1 \leq i \leq n$, such

that

$$\xi_0 = a < \xi_1 < \cdots < \xi_i < \cdots < \xi_{n-1} < \xi_n = b,$$
$$g(\xi) = c_i \quad \text{for all } \xi_{i-1} < \xi < \xi_i, \quad 1 \le i \le n,$$
$$\max_{0\le i\le n} \|g(\xi_i)\|_Y \le \max_{1\le j\le n} \|c_j\|_Y.$$

The integral of such a step function is then defined in the most natural way, i.e. by $\ell(g) := \sum_{i=1}^{n}(\xi_i - \xi_{i-1})c_i \in Y$. It is easily seen that the set $\mathcal{S}([a,b];Y)$ formed by all step functions over $[a,b]$ with values in Y is a vector space. Equipped with the *sup-norm*, defined by $\|g\| := \sup_{a\le\xi\le b} \|g(\xi)\|_Y$, the space $\mathcal{S}([a,b];Y)$ becomes a *normed vector space*. Then, the above mapping $\ell : \mathcal{S}([a,b];Y) \to Y$, which is clearly linear, is continuous over this space since

$$\|\ell(g)\|_Y \le (b-a)\|g\| \quad \text{for all } g \in \mathcal{S}([a,b];Y),$$

as the definition of $\ell(g)$ immediately shows.

Second, let $\mathcal{R}([a,b];Y)$ denote the *completion* of the space $\mathcal{S}([a,b];Y)$ with respect to the sup-norm $\|\cdot\|$, which is thus a *closed* subspace of the Banach space $\mathcal{B}([a,b];Y)$ of all *bounded* functions from $[a,b]$ into Y, again equipped with the same sup-norm $\|\cdot\|$. Then, the continuous linear mapping $\ell : \mathcal{S}([a,b];Y) \to Y$ admits a unique continuous extension to the space $\mathcal{R}([a,b];Y)$, since $\mathcal{S}([a,b];Y)$ is by construction *dense* in $\mathcal{R}([a,b];Y)$ and Y is *complete*; this is why the assumed completeness of Y is essential. Applying this extension to any function $g \in \mathcal{R}([a,b];Y)$ thus provides a natural *definition of the integral of* g *over* $[a,b]$, *denoted* $\int_a^b g(\xi)\,\mathrm{d}\xi$. Then, by construction, the vector $\int_a^b g(\xi)\,\mathrm{d}\xi \in Y$ defined in this fashion satisfies the inequality

$$\left\| \int_a^b g(\xi)\,\mathrm{d}\xi \right\|_Y \le (b-a)\|g\| \quad \text{for all } g \in \mathcal{R}([a,b];Y),$$

where $\|g\| = \sup_{a\le\xi\le b} \|g(\xi)\|_Y$.

Likewise, the inequality $\|\ell(g)\|_Y \le \int_a^b \|g(\xi)\|_Y\,\mathrm{d}\xi$, which clearly holds for all step functions $g \in \mathcal{S}([a,b];Y)$, also holds for all functions in the closure $\mathcal{R}([a,b];Y)$ of $\mathcal{S}([a,b];Y)$ since each side of this inequality is a continuous function of $g \in \mathcal{R}([a,b];Y)$. In other words,

$$\left\| \int_a^b g(\xi)\,\mathrm{d}\xi \right\|_Y \le \int_a^b \|g(\xi)\|_Y\,\mathrm{d}\xi \quad \text{for all } g \in \mathcal{R}([a,b];Y).$$

Let now $g : [a,b] \to Y$ be any *continuous* function. Since the interval $[a,b]$ is *compact*, the function g is then *uniformly continuous* over $[a,b]$, which easily implies that g *is a uniform limit of step functions* $g_n \in \mathcal{S}[a,b]$, $n \ge 1$. Hence, the integral $\int_a^b g(\xi)\,\mathrm{d}\xi \in Y$ is well defined, as $\lim_{n\to\infty} \ell(g_n)$ (as is well known, this limit is independent of the sequence of step functions chosen to approximate g).

Remark. The functions in the space $\mathcal{R}([a,b];Y)$ are called *regulated functions*. The above considerations thus show that the space $\mathcal{C}([a,b];Y)$ of all continuous functions with values in Y satisfy the inclusion $\mathcal{C}([a,b];Y) \subset \mathcal{R}([a,b];Y)$ (one can show that this inclusion is always *strict*; for instance, the space $\mathcal{R}([a,b];\mathbb{R})$ contains all monotone functions, which may be discontinuous at a countably infinite number of points of $[a,b]$).

2.4. *The mean value theorem for functions of class $\mathcal{C}^1$ with values in a Banach space*

The well-known *mean value theorem* for a continuously differentiable real function of a real variable asserts that $f(b) - f(a) = (b-a)\int_0^1 f'((1-\theta)a + \theta b)\,\mathrm{d}\theta$ (with self-explanatory notations). We now show how this classical theorem can be easily extended to functions that take their values in an arbitrary *Banach space*. This extension plays a key role in the proof of the Newton–Kantorovich theorem.

Note that the integral $\int_0^1 f'((1-\theta)a + \theta b)(b-a)\,\mathrm{d}\theta$ found in the next theorem makes sense since the function $\theta \in [0,1] \to f'((1-\theta)a+\theta b)(b-a) \in Y$ is continuous and Y is a Banach space (see Sec. 2.3).

Recall that a *closed segment* $[a,b]$ in a normed vector space is a set of the form $[a,b] = \{(1-\theta)a + \theta b; 0 \le \theta \le 1\}$.

Theorem 1 (Mean Value Theorem for Functions of Class $\mathcal{C}^1$ with Values in a Banach Space). *Let Ω be an open subset in a normed vector space X, let Y be a Banach space, and let $f \in \mathcal{C}^1(\Omega;Y)$. Then, given any closed segment $[a,b] \subset \Omega$,*

$$f(b) - f(a) = \int_0^1 f'((1-\theta)a + \theta b)(b-a)\,\mathrm{d}\theta.$$

Proof. Let I be an open interval of $\mathbb{R}$ containing the interval $[0,1]$. Given any function $g \in \mathcal{C}(\overline{I};Y)$, define the function

$$G : \theta \in I \to G(\theta) := \int_0^\theta g(\xi)\,\mathrm{d}\xi \in Y,$$

so that, given any point $\theta \in [0,1]$ and any $h > 0$ such that $(\theta + h) \in I$,

$$G(\theta+h) - G(\theta) - hg(\theta) = \int_\theta^{\theta+h} (g(\xi) - g(\theta))\,\mathrm{d}\xi.$$

Consequently,

$$\begin{aligned}\|G(\theta+h) - G(\theta) - hg(\theta)\|_Y &\le \int_\theta^{\theta+h} \|g(\xi) - g(\theta)\|_Y\,\mathrm{d}\xi \\ &\le h \sup_{\theta \le \xi \le \theta+h} \|g(\xi) - g(\theta)\|_Y,\end{aligned}$$

which in turn implies that

$$G(\theta+h) = G(\theta) + hg(\theta) + h\delta(h) \quad \text{with } \lim_{h \to 0^+} \delta(h) = 0 \text{ in } Y,$$

since the function g is continuous by assumption. A similar argument shows that the last relation also holds if $h < 0$, this time with $\lim_{h\to 0^-} \delta(h) = 0$. This shows that *the function $G : I \to Y$ is differentiable at each point of* $[0,1]$, with a derivative given by

$$G'(\theta) = g(\theta) \quad \text{in } Y \text{ at each } \theta \in [0,1]$$

(by definition of the Fréchet derivative, $G'(\theta) \in \mathcal{L}(\mathbb{R}; Y)$; but this relation makes sense as an equality in the space Y, since the space $\mathcal{L}(\mathbb{R}; Y)$ can be identified with Y).

Given a function $f \in \mathcal{C}^1(\Omega; Y)$ and a closed segment $[a,b] \subset \Omega$, there exists an open interval $I \subset \mathbb{R}$ containing $[0,1]$ such that $\{(1-\theta)a + \theta b; \theta \in \overline{I}\} \subset \Omega$ since Ω is open. Then, the function

$$g : \theta \in \overline{I} \to g(\theta) := f'((1-\theta)a + \theta b)(b-a) \in Y$$

belongs to the space $\mathcal{C}(\overline{I}; Y)$. Hence, by the above argument,

$$g(\theta) = G'(\theta) \quad \text{in } Y \text{ at each } \theta \in [0,1],$$

where $G(\theta) := \int_0^\theta g(\xi)\,\mathrm{d}\xi, 0 \le \theta \le 1$, on the one hand.

On the other hand, it is easily seen that the same function $g \in \mathcal{C}(\overline{I}; Y)$ satisfies

$$g(\theta) = \widetilde{G}'(\theta) \quad \text{in } Y \quad \text{at each } \theta \in [0,1],$$

where $\widetilde{G}(\theta) := f((1-\theta)a + \theta b) \in Y,\ 0 \le \theta \le 1$. Since the two functions G and $\widetilde{G}$ therefore share the same derivative at each point of the *connected* open interval $]0,1[$, they are equal on $]0,1[$, up to a constant vector in Y; hence, also on $[0,1]$ by continuity. There thus exists a vector $c \in Y$ such that $G(\theta) = \widetilde{G}(\theta) + c$ for all $0 \le \theta \le 1$. In particular then, $G(1) - G(0) = \widetilde{G}(1) - \widetilde{G}(0)$, or equivalently,

$$\int_0^1 f'((1-\theta)a + \theta b)(b-a)\,\mathrm{d}\theta = f(b) - f(a),$$

as was to be proved. □

The following useful consequence of Theorem 1 will be also needed in the proof of the Newton–Kantorovich theorem.

Theorem 2 (Corollary to the Mean Value Theorem). *The assumptions are those of Theorem* 1. *Then, given any continuous linear operator $A \in \mathcal{L}(X; Y)$, the following inequality holds*:

$$\|f(b) - f(a) - A(b-a)\|_Y \le \sup_{x\in[a,b]} \|f'(x) - A\|_{\mathcal{L}(X;Y)} \|b-a\|_X.$$

Proof. The mean value theorem applied to the function $g : x \in \Omega \to g(x) := (f(x) - Ax) \in Y$, whose derivative at $x \in \Omega$ is $g'(x) = (f'(x) - A) \in \mathcal{L}(X; Y)$, gives

$$g(b) - g(y) = f(b) - f(a) - A(b-a) = \int_0^1 (f'((1-\theta)a + \theta b) - A)(b-a)\,\mathrm{d}\theta.$$

Consequently, since $\|\int_0^1 h(\theta)\,\mathrm{d}\theta\|_Y \le \int_0^1 \|h(\theta)\|_Y\,\mathrm{d}\theta$ for all $h \in \mathcal{C}([a,b];Y)$ (Sec. 2.3),

$$\|f(b)-f(a)-A(b-a)\|_Y \le \int_0^1 \|f'((1-\theta)a+\theta b)-A\|_{\mathcal{L}(X;Y)}\|b-a\|_X\,\mathrm{d}\theta,$$

from which the announced inequality clearly follows. □

3. The Classical Newton–Kantorovich Theorem "with Three Constants"

The following result is a *basic theorem of nonlinear functional analysis*, as well as a *basic theorem of numerical analysis*.

This theorem is due to Kantorovich [7]. A different proof was later given in Kantorovich and Akilov [8]. The proof given here of the convergence of the sequence of Newton iterates, which follows the latter but is simpler, is adapted from the illuminating, but concise, treatment of Ortega [10]. It is similar to the proofs given in Deimling [3] and Zeidler [13], although more detailed, and more complete regarding the uniqueness issue. Interesting complements and in-depth treatments are found in Rheinboldt [11], Gragg and Tapia [6], Deuflhard [4], Dedieu [2], among others (this brief list is by no means intended to be exhaustive).

Theorem 3 (The Classical Newton–Kantorovich Theorem "with Three Constants"). *Let there be given two Banach spaces X and Y, an open subset Ω of X, a point $x_0 \in \Omega$, and a mapping $f \in \mathcal{C}^1(\Omega;Y)$ such that $f'(x_0) \in \mathcal{L}(X;Y)$ is one-to-one and onto, so that $f'(x_0)^{-1} \in \mathcal{L}(Y;X)$ (by Banach open mapping theorem; see, e.g., [1, 9], or [12]). Assume that there exist three constants λ, μ, ν such that*

$$0 < \lambda\mu\nu \le \frac{1}{2} \quad \text{and} \quad B(x_0;r) \subset \Omega, \quad \text{where } r := \frac{1}{\mu\nu},$$

$$\|f'(x_0)^{-1}f(x_0)\|_X \le \lambda,$$

$$\|f'(x_0)^{-1}\|_{\mathcal{L}(Y;X)} \le \mu,$$

$$\|f'(\widetilde{x})-f'(x)\|_{\mathcal{L}(X;Y)} \le \nu\|\widetilde{x}-x\|_X \quad \text{for all } \widetilde{x}, x \in B(x_0;r).$$

Then $f'(x) \in \mathcal{L}(X;Y)$ is one-to-one and onto and $f'(x)^{-1} \in \mathcal{L}(Y;X)$ at each $x \in B(x_0;r)$, the sequence $(x_k)_{k=0}^{\infty}$ defined by

$$x_{k+1} = x_k - f'(x_k)^{-1}f(x_k), \quad k \ge 0,$$

is contained in the ball $B(x_0;r_-)$, where

$$r_- := \frac{1-\sqrt{1-2\lambda\mu\nu}}{\mu\nu} \le r,$$

and converges to a zero $a \in \overline{B(x_0;r_-)}$ of f. Besides, for each $k \ge 0$,

$$\|x_k-a\|_X \le \frac{r}{2^k}\left(\frac{r_-}{r}\right)^{2^k} \quad \text{if } \lambda\mu\nu < \frac{1}{2}, \quad \text{or} \quad \|x_k-a\|_X \le \frac{r}{2^k} \quad \text{if } \lambda\mu\nu = \frac{1}{2}.$$

If $\lambda\mu\nu < \frac{1}{2}$, *assume in addition that*

$$B(x_0; r_+) \subset \Omega, \quad \text{where } r_+ := \frac{1+\sqrt{1-2\lambda\mu\nu}}{\mu\nu},$$

$$\|f'(\widetilde{x}) - f'(x)\|_{\mathcal{L}(X;Y)} \leq \nu\|\widetilde{x} - x\|_X \quad \text{for all } \widetilde{x}, x \in B(x_0; r_+).$$

Then, the point $a \in \overline{B(x_0; r_-)}$ *is the only zero of* f *in* $B(x_0; r_+)$.

If $\lambda\mu\nu = \frac{1}{2}$ (*in which case* $r_- = r = r_+$), *assume in addition that*

$$\overline{B(x_0; r)} \subset \Omega.$$

Then, the point $a \in \overline{B(x_0; r)}$ *is the only zero of* f *in* $\overline{B(x_0; r)}$.

Proof. Let the numbers $t_k, k \geq 0$, with $t_0 := 0$, be the *Newton iterates for the quadratic polynomial* (the specific form of which can be *a priori* justified; see part (vii) of the proof)

$$p : t \in \mathbb{R} \to p(t) := \frac{\mu\nu}{2}t^2 - t + \lambda.$$

The key idea of the proof rests on the *majorant method*, which consists in showing that *the sequence* $(t_k)_{k=0}^{\infty}$ *majorizes the sequence* $(x_k)_{k=0}^{\infty}$ *formed by the Newton iterates* $x_{k+1} = x_k - f'(x_k)^{-1}f(x_k), k \geq 0$, in the sense that

$$\|x_{k+1} - x_k\|_X \leq t_{k+1} - t_k \quad \text{for all } k \geq 0.$$

This property will in turn imply that the sequence $(x_k)_{k=0}^{\infty}$ converges to a zero a of f and that

$$\|x_k - a\|_X \leq r_- - t_k \quad \text{for all } k \geq 0,$$

where $r_- = \lim_{k\to\infty} t_k$ is the smallest root of p. This explains why the proof begins by an analysis of the behavior of the Newton iterates $t_k, k \geq 0$, for the polynomial p.

For convenience, the proof is divided into eight parts, numbered (i) to (viii).

(i) *The Newton iterates*

$$t_0 := 0 \quad \text{and} \quad t_{k+1} := t_k - \frac{p(t_k)}{p'(t_k)} = t_k + \frac{\frac{\mu\nu}{2}t_k^2 - t_k + \lambda}{1 - \mu\nu t_k}, \quad k \geq 0,$$

for the polynomial p *satisfy the following relations*

$$t_{k+1} - t_k = \frac{\mu\nu(t_k - t_{k-1})^2}{2(1-\mu\nu t_k)}, \quad k \geq 1,$$

$$r_- - t_{k+1} = \frac{\mu\nu(r_- - t_k)^2}{2(1-\mu\nu t_k)}, \quad k \geq 0,$$

$$t_{k+1} - t_k \le \frac{\lambda}{2^k}, \quad k \ge 0,$$

$$1 - \mu\nu t_k \ge \frac{1}{2^k}, \quad k \ge 0,$$

$$r_- - t_k \le \frac{1}{\mu\nu 2^k}(\mu\nu r_-)^{2^k}, \quad k \ge 0,$$

where $r_- := \frac{1-\sqrt{1-2\lambda\mu\nu}}{\mu\nu}$ *is the smallest root of* p *if* $\lambda\mu\nu < \frac{1}{2}$ (in which case $\mu\nu r_- < 1$) *and* $r_- = r := \frac{1}{\mu\nu}$ *is the double root of* p *if* $\lambda\mu\nu = \frac{1}{2}$ (in which case $\mu\nu r_- = 1$).

The proofs of these relations and estimates rely on straightforward computations, which are left to the reader.

For brevity, all norms will be denoted by the same symbol $\|\cdot\|$ in the rest of the proof.

(ii) *A first functional analytic preliminary*: *The mapping* $f : \Omega \subset X \to Y$ *satisfies*

$$\|f(\widetilde{x}) - f(x) - f'(x)(\widetilde{x} - x)\| \le \frac{\nu}{2}\|\widetilde{x} - x\|^2 \quad \text{for all } \widetilde{x}, x \in B(x_0; r).$$

The proof of this inequality rests on the *mean value theorem for functions of class* $\mathcal{C}^1$ *with values in a Banach space* (Theorem 1), applied to the function $f \in \mathcal{C}^1(\Omega; Y)$ between any two points x and $\widetilde{x}$ in the open subset $B(x_0; r)$ of Ω (as a convex set, the ball $B(x_0; r)$ contains the closed segment $[x, \widetilde{x}]$). This gives

$$f(\widetilde{x}) - f(x) = \int_0^1 f'((1-\theta)x + \theta\widetilde{x})(\widetilde{x} - x)\,\mathrm{d}\theta \quad \text{for all } \widetilde{x}, x \in B(x_0; r).$$

Noting that the expression $f(\widetilde{x}) - f(x) - f'(x)(\widetilde{x} - x)$ can be also written as

$$f(\widetilde{x}) - f(x) - f'(x)(\widetilde{x} - x) = \int_0^1 (f'((1-\theta)x + \theta\widetilde{x}) - f'(x))(\widetilde{x} - x)\,\mathrm{d}\theta,$$

we conclude that

$$\begin{aligned}\|f(\widetilde{x}) - f(x) - f'(x)(\widetilde{x} - x)\| &\le \left(\int_0^1 \|f'((1-\theta)x + \theta\widetilde{x}) - f'(x)\|\,\|\widetilde{x} - x\|\,\mathrm{d}\theta\right) \\ &\le \int_0^1 \nu\theta\,\|\widetilde{x} - x\|^2\,\mathrm{d}\theta = \frac{\nu}{2}\|\widetilde{x} - x\|^2.\end{aligned}$$

(iii) *A second functional analytic preliminary*: *Given any* $x \in B(x_0; r)$, *the derivative* $f'(x) \in \mathcal{L}(X; Y)$ *is one-to-one and onto, and* $f'(x)^{-1} \in \mathcal{L}(Y; X)$. *Besides*,

$$\|f'(x)^{-1}\| \le \frac{\mu}{1 - \mu\nu\|x - x_0\|} \quad \text{for all } x \in B(x_0; r).$$

Noting that

$$\|x - x_0\| < \frac{1}{\mu\nu} \quad \text{implies } \|f'(x_0)^{-1}(f'(x) - f'(x_0))\| \le \mu\nu\|x - x_0\| < 1,$$

we infer that, if $x \in B(x_0;r)$, then $f'(x) \in \mathcal{L}(X;Y)$ is one-to-one and onto and $f'(x)^{-1} \in \mathcal{L}(Y;X)$, with (Sec. 2.1)

$$\|f'(x)^{-1}\| \le \frac{\|f'(x_0)^{-1}\|}{1-\|f'(x_0)^{-1}(f'(x)-f'(x_0))\|} \le \frac{\mu}{1-\mu\nu\|x-x_0\|}.$$

(iv) *A third — and last — functional analytic preliminary*: *Define the auxiliary function*

$$g : x \in B(x_0;r) \to g(x) := x - f'(x)^{-1}f(x) \in X$$

(which is unambiguously defined by (iii)). *Then, given any* $x \in B(x_0;r)$ *such that* $g(x) \in B(x_0;r)$, *the following estimate holds*:

$$\|g(g(x)) - g(x)\| \le \frac{\mu\nu\|g(x)-x\|^2}{2(1-\mu\nu\|g(x)-x_0\|)}.$$

The estimate of (iii) shows that, given any $x \in B(x_0;r)$ such that $g(x) \in B(x_0;r)$,

$$\|g(g(x)) - g(x)\| = \|(f'(g(x)))^{-1}f(g(x))\| \le \frac{\mu\|f(g(x))\|}{1-\mu\nu\|g(x)-x_0\|}.$$

Noting that $f(x) + f'(x)(g(x)-x) = 0$ for all $x \in B(x_0;r)$ by definition of the function g, we infer from (ii) (which can be applied, since both x and $g(x)$ belong to $B(x_0;r)$) that

$$\|f(g(x))\| = \|f(g(x))-f(x)-f'(x)(g(x)-x)\| \le \frac{\nu}{2}\|g(x)-x\|^2 \quad \text{for all } x \in B(x_0;r).$$

Hence the announced estimate holds.

(v) *The Newton iterates* $x_{k+1} := x_k - (f'(x_k))^{-1}f(x_k), k \ge 0$, *for the mapping* f *belong to the open ball* $B(x_0;r_-)$ (hence, they are well defined) *and they satisfy the estimate*

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k \quad \text{for all } k \ge 0,$$

where the numbers $t_k, k \ge 0$, *are the Newton iterates for the polynomial* $p : t \in \mathbb{R} \to \frac{\mu\nu}{2}t^2 - t + \lambda$ *when* $t_0 = 0$ (see (i)).

The announced properties hold for $k = 0$ since

$$\|x_1 - x_0\| = \|f'(x_0)^{-1}f(x_0)\| \le \lambda = t_1 < r_-.$$

So, assume that they hold for $k = 0, \dots, n-1$, for some integer $n \ge 1$, so that

$$\|x_n - x_0\| \le \sum_{\ell=0}^{n-1}\|x_{\ell+1} - x_\ell\| \le \sum_{\ell=0}^{n-1}(t_{\ell+1} - t_\ell) = t_n - t_0 = t_n.$$

Then,

$$x_{n+1} := x_n - (f'(x_n))^{-1}f(x_n) = g(x_n)$$

is well defined (since $x_n \in B(x_0; r_-)$ by the induction hypothesis and thus $(f'(x_n))^{-1} \in \mathcal{L}(Y; X)$ is well defined; cf. (iii)). We thus have

$$x_{n+1} - x_n = g(x_n) - g(x_{n-1}) = g(g(x_{n-1})) - g(x_{n-1}),$$

so that, by (iv) (which can be applied since both x_{n-1} and $g(x_{n-1}) = x_n$ belong to $B(x_0; r_-)$ by the induction hypothesis) and (i),

$$\begin{aligned}\|x_{n+1} - x_n\| = \|g(g(x_{n-1})) - g(x_{n-1})\| &\le \frac{\mu\nu\|g(x_{n-1}) - x_{n-1}\|^2}{2(1 - \mu\nu\|g(x_{n-1}) - x_0\|)} \\ &= \frac{\mu\nu\|x_n - x_{n-1}\|^2}{2(1 - \mu\nu\|x_n - x_0\|)} \le \frac{\mu\nu(t_n - t_{n-1})^2}{2(1 - \mu\nu t_n)} = t_{n+1} - t_n.\end{aligned}$$

Finally,

$$\|x_{n+1} - x_0\| \le \sum_{\ell=0}^{n} \|x_{\ell+1} - x_\ell\| \le t_{n+1} < r_-.$$

Hence, the announced properties hold for $k = n$.

(vi) *The Newton iterates* $x_k \in B(x_0; r_-), k \ge 0$, *converge to a zero* $a \in \overline{B(x_0; r_-)}$ *of* f, *and*

$$\|a - x_k\| \le \frac{1}{\mu\nu 2^k}(\mu\nu r_-)^{2^k}, \quad k \ge 0.$$

Since $\|x_m - x_n\| \le \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \le t_m - t_n$ for all $m > n \ge 0$ and the sequence $(t_k)_{k=0}^{\infty}$ converges as $k \to \infty$ (to r_-), the sequence $(x_k)_{k=0}^{\infty}$ is a Cauchy sequence in the closed ball $\overline{B(x_0; r_-)}$, which is a complete metric space (as a closed subset of the Banach space X). Hence, *the sequence* $(x_k)_{k=0}^{\infty}$ *converges to a point* $a \in \overline{B(x_0; r_-)}$. Besides,

$$\begin{aligned}\|f(x_k)\| = \|f'(x_k)(x_{k+1} - x_k)\| &\le (\|f'(x_0)\| + \|f'(x_k) - f'(x_0)\|)\,\|x_{k+1} - x_k\| \\ &\le (\|f'(x_0)\| + \nu\|x_k - x_0\|)\,\|x_{k+1} - x_k\| \le (\|f'(x_0)\| + \nu r_-)(t_{k+1} - t_k), \\ &\qquad\qquad k \ge 0.\end{aligned}$$

Consequently, $f(a) = \lim_{k\to\infty} f(x_k) = 0$ (the function f is continuous in $\overline{B(x_0; r_-)}$ since it is differentiable there by assumption). Hence, *a is a zero of* f.

Letting $\ell \to \infty$ in the inequality $\|x_\ell - x_k\| \le t_\ell - t_k$ further shows that

$$\|a - x_k\| \le r_- - t_k \quad \text{for each } k \ge 0.$$

Hence the announced estimate for $\|a - x_k\|$ follows from (i).

(vii) *Uniqueness of a zero of* f *in* $B(x_0; r_+)$ *when* $\lambda\mu\nu < \frac{1}{2}$ *under the additional assumptions that* $B(x_0; r_+) \subset \Omega$ *and*

$$\|f'(\widetilde{x}) - f'(x)\| \le \nu\|\widetilde{x} - x\| \quad \text{for all } \widetilde{x}, x \in B(x_0; r_+).$$

Define the auxiliary function

$$h : x \in \Omega \to h(x) := f'(x_0)^{-1} f(x) \in X,$$

whose zeros are thus the same as those of the function f. Clearly then, $h \in \mathcal{C}^1(\Omega; X)$ and the derivative of h at each $x \in \Omega$ is given by $h'(x) = f'(x_0)^{-1} f'(x)$, so that in particular

$$h'(x_0) = \mathrm{id}_X,$$

where id_X denotes the identity mapping in X, and

$$\|h'(\widetilde{x}) - h'(x)\| \leq \|f'(x_0)^{-1}\| \|f'(\widetilde{x}) - f'(x)\| \leq \mu\nu \|\widetilde{x} - x\| \quad \text{for all } \widetilde{x}, x \in B(x_0; r_+).$$

First, we show that, if $\lambda\mu\nu \leq \frac{1}{2}$, *the function f has at most one zero in the open ball $B(x_0; r)$*.

To this end, assume that $a, b \in B(x_0; r)$ are such that $f(a) = f(b) = 0$. Then, by the corollary to the mean value theorem (Theorem 2),

$$\|b - a\| = \|h(b) - h(a) - (b - a)\| \leq \left(\sup_{x \in [a,b]} \|h'(x) - \mathrm{id}_X\| \right) \|b - a\|.$$

Besides,

$$\sup_{x \in [a,b]} \|h'(x) - \mathrm{id}_X\| = \sup_{x \in [a,b]} \|h'(x) - h'(x_0)\| \leq \mu\nu \sup_{x \in [a,b]} \|x - x_0\| < \mu\nu r,$$

since

$$\begin{aligned} \sup_{x \in [a,b]} \|x - x_0\| &= \sup_{t \in [0,1]} \|(1 - t)(a - x_0) + t(b - x_0)\| \\ &\leq \max\{\|a - x_0\|, \|b - x_0\|\} < r. \end{aligned}$$

But $\mu\nu r = 1$; hence, $a = b$.

Second, we show that, *if $\lambda\mu\nu < \frac{1}{2}$, the function f does not have any zero in the set $B(x_0; r_+) - \overline{B(x_0; r_-)}$*. To this end, we infer from (ii) that

$$\|h(x) - h(x_0) - h'(x_0)(x - x_0)\| \leq \frac{\mu\nu}{2} \|x - x_0\|^2 \quad \text{for all } x \in B(x_0; r_+).$$

But $h'(x_0) = \mathrm{id}_X$ and $\|h(x_0)\| \leq \lambda$; hence,

$$\begin{aligned} \|h(x)\| &\geq \|h(x_0) + h'(x_0)(x - x_0)\| - \frac{\mu\nu}{2} \|x - x_0\|^2 \\ &\geq \|x - x_0\| - \|h(x_0)\| - \frac{\mu\nu}{2} \|x - x_0\|^2 \\ &\geq -\left(\frac{\mu\nu}{2} \|x - x_0\|^2 - \|x - x_0\| + \lambda \right) \\ &= -p(\|x - x_0\|) \quad \text{for all } x \in B(x_0, r_+). \end{aligned}$$

Since $p(t) < 0$ for all $r_- < t < r_+$ when $\lambda\mu\nu < \frac{1}{2}$, it follows that

$$\|h(x)\| > 0 \quad \text{for all } r_- < \|x - x_0\| < r_+.$$

Consequently, $f(x) \neq 0$ for all $x \in B(x_0; r_+) - \overline{B(x_0; r_-)}$, on the one hand. Since, on the other hand, f has at most one zero in $B(x_0; r)$, *the zero* $a \in \overline{B(x_0; r_-)}$ *found in* (vi) *is the only zero of* f *in* $B(x_0; r_+)$ *if* $\lambda\mu\nu < \frac{1}{2}$.

If $\lambda\mu\nu = \frac{1}{2}$, the preceding analysis only shows that, if it so happens that the zero a found in (vi) belongs to the *open* ball $B(x_0; r)$, then a is the only zero of f in this open ball; but no conclusion about uniqueness can be reached if $a \in \partial B(x_0; r)$. This is why this case is treated separately, in the next — and last — step of this proof.

(viii) *Uniqueness of a zero of* f *in* $\overline{B(x_0; r)}$ *when* $\lambda\mu\nu = \frac{1}{2}$, *under the additional assumption that* $\overline{B(x_0; r)} \subset \Omega$.

First, we notice that

$$\|f'(\widetilde{x}) - f'(x)\| \leq \nu\|\widetilde{x} - x\| \quad \text{for all } \widetilde{x}, x \in \overline{B(x_0; r)},$$

since this inequality, which holds by assumption for all $\widetilde{x}, x \in B(x_0; r)$, can be extended by continuity to $\overline{B(x_0; r)}$ if $\overline{B(x_0; r)} \subset \Omega$.

Our objective is to show that, when $\lambda\mu\nu = \frac{1}{2}$, the zero $a \in \overline{B(x_0; r)}$ found in (vi) is the only zero of f in $\overline{B(x_0; r)}$. To this end, we establish that, *if any point* $b \in \overline{B(x_0; r)}$ *satisfies* $f(b) = 0$ *when* $\lambda\mu\nu = \frac{1}{2}$, *then the Newton iterates* $x_{k+1} = x_k - f'(x_k)^{-1} f(x_k), k \geq 0$, *satisfy*

$$\|b - x_k\| \leq \frac{r}{2^k} \quad \text{for all } k \geq 0.$$

Clearly, this relation holds for $k = 0$; so, assume that it holds for $k = 0, \ldots, n$ for some integer $n \geq 0$. Since $f(b) = 0$ we may write $\|b - x_{n+1}\|$ as

$$\|b - x_{n+1}\| = \|f'(x_n)^{-1}(f(b) - f(x_n) - f'(x_n)(b - x_n))\|,$$

so that, from (ii) and the induction hypothesis,

$$\begin{aligned}\|b - x_{n+1}\| &\leq \|f'(x_n)^{-1}\|\|f(b) - f(x_n) - f'(x_n)(b - x_n)\| \\ &\leq \frac{\nu}{2}\|f'(x_n)^{-1}\|\|b - x_n\|^2 \leq \frac{\nu r^2}{2^{2n+1}}\|(f'(x_n))^{-1}\|.\end{aligned}$$

Besides, the inequality established in (iii) shows that, in particular,

$$\|f'(x_n)^{-1}\| \leq \frac{\mu}{1 - \mu\nu\|x_n - x_0\|}.$$

Recalling that $t_0 = 0$ and $t_{k+1} - t_k \leq \frac{\lambda}{2^k}, k \geq 0$, and that $\|x_n - x_0\| \leq t_n$ (see (i) and (v)), we next infer that

$$\|x_n - x_0\| \leq t_n \leq \lambda\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) = 2\lambda\left(1 - \frac{1}{2^n}\right).$$

Therefore,

$$\|b - x_{n+1}\| \leq \left(\frac{\mu\nu r}{(1 - 2\lambda\mu\nu(1 - 2^{-n}))2^n} \right) \frac{r}{2^{n+1}} = \frac{r}{2^{n+1}},$$

since $\mu\nu r = 2\lambda\mu\nu = 1$. Hence,

$$\|b - x_k\| \leq \frac{r}{2^k} \quad \text{for all } k \geq 0.$$

Consequently,

$$\|b - a\| = \lim_{k \to \infty} \|b - x_k\| = 0,$$

which shows that $b = a$. This completes the proof. □

Remark. The inequality $\|f'(x_0)^{-1} f(x)\| \geq -p(\|x - x_0\|)$ for all $x \in \overline{B(x_0; r)}$ established in part (vii) of the above proof provides a motivation for the explicit form of the polynomial p.

The assumptions and conclusions of Theorem 3 perfectly display the *strength* and *weakness* of Newton's method. Its strength is the *speed of convergence*, as it provides rates of convergence that are *quadratic if* $\lambda\mu\nu = \frac{1}{2}$ (as reflected by the factor $\frac{1}{2^k}$ in the error estimate), and *faster than quadratic if* $\lambda\mu\nu < \frac{1}{2}$ (as reflected by the factor $\frac{1}{2^k}(\frac{r_-}{r})^{2^k}$ in the error estimate); for instance, if α is a "moderately large" number, say $10 \leq \alpha \leq 500$, the iterative method $x_{k+1} = \frac{1}{2}(x_k + \alpha/x_k), k \geq 0$, with $x_0 = 10$ already provides an approximation of $\sqrt{\alpha}$ that is correct up to 10 decimals after only six iterations! Its weakness is its *sensibility to the choice of the initial guess* x_0, as its success for finding a zero is highly dependent on its "closeness to a zero", expressed in a somewhat subtle way by the assumption $\lambda\mu\nu \leq \frac{1}{2}$.

As an example of application, consider the *nonlinear two-point boundary value problem*:

$$-u''(t) + u(t)^p = \varphi(t), \quad 0 \leq t \leq 1,$$
$$u(0) = u(1) = 0,$$

where $p \geq 2$ is an integer and $\varphi \in \mathcal{C}[0, 1]$ is a given function. Note that the following results apply as well to the problem $-u''(t) - u(t)^p = \varphi(t), 0 \leq t \leq 1$, and $u(0) = u(1) = 0$.

It is well known that finding a solution $u \in \mathcal{C}^2[0, 1]$ to such a boundary value problem is the same as finding a solution $u \in \mathcal{C}[0, 1]$ to a *nonlinear integral equation*, which in this case takes the form

$$u(t) = \int_0^1 G(t, \xi)(\varphi(\xi) - u(\xi)^p)\, \mathrm{d}\xi, \quad 0 \leq t \leq 1,$$

the function G being defined by $G(t, \xi) := \xi(1 - t)$ if $0 \leq \xi \leq t \leq 1$ and $G(t, \xi) := t(1 - \xi)$ if $0 \leq t < \xi \leq 1$. Solving this integral equation in turn amounts to *finding a zero of the nonlinear mapping* $f : u \in \mathcal{C}[0, 1] \to f(u) \in \mathcal{C}[0, 1]$ *defined by*

$$(f(u))(t) = u(t) + \int_0^1 G(t, \xi)(u(\xi)^p - \varphi(\xi))\, \mathrm{d}\xi, \quad 0 \leq t \leq 1.$$

In what follows, the space

$$X := \mathcal{C}[0,1]$$

is equipped with the sup-norm, denoted $\|\cdot\|_X$, which makes it a *Banach space*. Then, it is easily seen that the mapping f is of class $\mathcal{C}^1$, with a Fréchet derivative $f'(u) \in \mathcal{L}(X)$ (Sec. 2.2) given by

$$f'(u)v = v + p\int_0^1 G(\cdot,\xi)u(\xi)^{p-1}v(\xi)\,\mathrm{d}\xi \quad \text{for all } v \in X.$$

Let u_0 denote the function equal to zero on $[0,1]$. Then, straightforward computations show that

$$\|f'(u_0)^{-1}f(u_0)\|_X = \frac{1}{8}\|\varphi\|_X, \quad \|f'(u_0)^{-1}\|_{\mathcal{L}(X)} = 1,$$

$$\|f'(\widetilde{u}) - f'(u)\|_{\mathcal{L}(X)} \le \frac{1}{8}p(p-1)r^{p-2}\|\widetilde{u} - u\|_X \quad \text{for all } \widetilde{u}, u \in B(u_0;r)$$

and any $r > 0$,

and that, if $\|\varphi\|_X \le 4r_p$, where $r_p := (\frac{8}{p(p-1)})^{\frac{1}{p-1}}$, the assumptions of the Newton–Kantorovich theorem are satisfied. This shows that, in this case, *the nonlinear two-point boundary value problem has a solution* and that *this solution can be approximated by Newton's method*. More specifically, given the kth Newton iterate u_k, finding the $(k+1)$th iterate u_{k+1} amounts to solving the *linear* boundary value problem:

$$-u''(t) + p(u_k(t))^{p-1}u(t) = (p-1)(u_k(t))^p - \varphi(t), \quad 0 \le t \le 1,$$

$$u(0) = u(1) = 0.$$

Remark. One of the most powerful existence theorems for a nonlinear boundary value problem of the form $-u''(t) + f(t,u(t)) = 0, 0 \le t \le 1$, and $u(0) = u(1) = 0$ asserts that it has a solution if there exists a constant c such that $\frac{\partial f}{\partial u}(t,v) \ge c > -\pi^2$ for all $0 \le t \le 1$ and $v \in \mathbb{R}$ (see, e.g., [14]). The above example thus illustrates the power of the Newton–Kantorovich theorem, seen here as an efficient alternative for proving existence theorems (the above sufficient condition is certainly not satisfied if the exponent p is even).

4. The Classical Newton–Kantorovich Theorem "with Only Two Constants"

We now show how the number of constants appearing in the assumptions of the classical Newton–Kantorovich theorem can be reduced from three to two, thanks to a very simple change in the formulation of the assumptions.

Theorem 4 (Newton–Kantorovich Theorem "with Only Two Constants"). *Let there be given two Banach spaces X and Y, an open subset Ω of X, a point $x_0 \in \Omega$, and a mapping $f \in \mathcal{C}^1(\Omega;Y)$ such that $f'(x_0) \in \mathcal{L}(X;Y)$ is one-to-one*

and onto (hence, $f'(x_0)^{-1} \in \mathcal{L}(Y;X)$). Assume that there exist two constants λ and r such that

$$0 < \lambda \leq \frac{r}{2} \quad \textit{and} \quad B(x_0;r) \subset \Omega,$$

$$\|f'(x_0)^{-1}f(x_0)\|_X \leq \lambda,$$

$$\|f'(x_0)^{-1}(f(\widetilde{x}) - f'(x))\|_{\mathcal{L}(X)} \leq \frac{1}{r}\|\widetilde{x} - x\|_X \quad \textit{for all } \widetilde{x}, x \in B(x_0;r).$$

Then, $f'(x) \in \mathcal{L}(X;Y)$ is one-to-one and onto and $f'(x)^{-1} \in \mathcal{L}(Y;X)$ at each $x \in B(x_0;r)$. The sequence $(x_k)_{k=0}^{\infty}$ defined by

$$x_{k+1} = x_k - f'(x_k)^{-1}f(x_k), \quad k \geq 0,$$

is such that

$$x_k \in B(x_0;r_-) \quad \textit{for all } k \geq 0, \quad \textit{where } r_- := r\left(1 - \sqrt{1 - \frac{2\lambda}{r}}\right) \leq r,$$

and converges to a zero $a \in \overline{B(x_0;r_-)}$ of f. Besides, for each $k \geq 0$,

$$\|x_k - a\| \leq \frac{r}{2^k}\left(\frac{r_-}{r}\right)^{2^k} \quad \textit{if } \ 0 < \lambda < \frac{r}{2}, \quad \textit{or} \quad \|x_k - a\| \leq \frac{r}{2^k} \quad \textit{if } \lambda = \frac{r}{2}.$$

If $0 < \lambda < \dfrac{r}{2}$, assume in addition that

$$B(x_0;r_+) \subset \Omega, \quad \textit{where } r_+ := r\left(1 + \sqrt{1 - \frac{2\lambda}{r}}\right),$$

$$\|f'(x_0)^{-1}(f(\widetilde{x}) - f'(x))\|_{\mathcal{L}(X)} \leq \frac{1}{r}\|\widetilde{x} - x\|_X \quad \textit{for all } \ \widetilde{x}, x \in B(x_0;r_+).$$

Then, the point $a \in \overline{B(x_0;r_-)}$ is the only zero of f in $B(x_0;r_+)$.

If $\lambda = \dfrac{r}{2}$ (in which case $r_- = r = r_+$), assume, in addition, that $\overline{B(x_0;r)} \subset \Omega$. Then, the point $a \in \overline{B(x_0;r)}$ is the only zero of f in $\overline{B(x_0;r)}$.

Proof. Rather than adapting step-by-step the proof of Theorem 3 under these new assumptions, it is much quicker to use the following simple observation: With the same notations and assumptions as in Theorem 3, define (as in part (vii) of the proof of Theorem 3) the auxiliary function $h \in \mathcal{C}^1(\Omega;X)$ by

$$h(x) := f'(x_0)^{-1}f(x), \quad x \in \Omega,$$

so that $h'(x) = f'(x_0)^{-1}f'(x), x \in \Omega$. *Then, the Newton iterates for the mapping h coincide with those for the mapping f* since

$$x_{k+1} - x_k = -h'(x_k)^{-1}h(x_k) = -f'(x_k)^{-1}f(x_k), \quad k \geq 0.$$

It thus suffices to check that the assumptions of Theorem 3 hold for the function h (instead of the function f). Since in this case we can choose

$$\mu := \|h'(x_0)^{-1}\| = \|\mathrm{id}_X\| = 1,$$

these assumptions are therefore satisfied if there exist two constants λ and ν such that

$$0 < \lambda\nu \leq \frac{1}{2} \quad \text{and} \quad B(x_0; r) \subset \Omega, \quad \text{where } r := \frac{1}{\nu},$$

$$\|h(x_0)\| = \|f'(x_0)^{-1} f(x_0)\| \leq \lambda,$$

$$\|h'(\widetilde{x}) - h'(x)\| = \|f'(x_0)^{-1}(f'(\widetilde{x}) - f'(x))\| \leq \nu\|\widetilde{x} - x\| \quad \text{for all } \widetilde{x}, x \in B(x_0; r),$$

which are precisely the assumptions made in Theorem 4. □

5. A Newton–Kantorovich Theorem "with Only One Constant"

We now give a substantially simpler statement (in that only one constant is needed in its assumptions) and a substantially simpler proof, both new to the best of our knowledge, of the Newton–Kantorovich theorem when $\lambda = \frac{r}{2}$. *The advantage of this new proof over the traditional proof is that it altogether avoids the Newton iterates* $t_k, k \geq 0$, *for the quadratic polynomial* p.

Its only drawback is that it does not yield the improved error estimates $\|x_k - a\| \leq \frac{r}{2^k}(\frac{r_-}{r})^{2^k}$ that holds when $\lambda < \frac{r}{2}$ (indeed, it seems that the Newton iterates $t_k, k \geq 0$, used in the majorant method seem unavoidable in order to obtain such improved error estimates when $\lambda < \frac{r}{2}$). But, in our opinion, this shortcoming is more than compensated by the simplicity of the proof.

Note that, like that of the classical Newton–Kantorovich theorem (Theorem 3), the proof of Theorem 5 is *self-contained*.

Theorem 5 (Newton–Kantorovich Theorem "with Only One Constant"). *Let there be given two Banach spaces* X *and* Y, *an open subset* Ω *of* X, *a point* $x_0 \in \Omega$, *and a mapping* $f \in \mathcal{C}^1(\Omega; Y)$ *such that* $f'(x_0) \in \mathcal{L}(X; Y)$ *is one-to-one and onto (hence,* $f'(x_0)^{-1} \in \mathcal{L}(X; Y)$*). Assume that there exists a constant* r *such that*

$$r > 0 \quad \textit{and} \quad \overline{B(x_0; r)} \subset \Omega,$$

$$\|f'(x_0)^{-1} f(x_0)\|_X \leq \frac{r}{2},$$

$$\|f'(x_0)^{-1}(f'(\widetilde{x}) - f'(x))\|_{\mathcal{L}(X)} \leq \frac{1}{r}\|\widetilde{x} - x\|_X \quad \textit{for all } \widetilde{x}, x \in B(x_0; r).$$

Then, $f'(x) \in \mathcal{L}(X; Y)$ *is one-to-one and onto and* $f'(x)^{-1} \in \mathcal{L}(Y; X)$ *at each* $x \in B(x_0; r)$. *The sequence* $(x_k)_{k=0}^{\infty}$ *defined by*

$$x_{k+1} = x_k - (f'(x_k))^{-1} f(x_k), \quad k \geq 0,$$

is such that $x_k \in B(x_0; r)$ *for all* $k \geq 0$ *and converges to a zero* $a \in \overline{B(x_0; r)}$ *of* f. *Besides, for each* $k \geq 0$,

$$\|x_k - a\| \leq \frac{r}{2^k},$$

and the point $a \in \overline{B(x_0; r)}$ *is the only zero of* f *in* $\overline{B(x_0; r)}$.

Proof. Like in the proofs of Theorems 3 and 4, we introduce the auxiliary function $h \in \mathcal{C}^1(\Omega;X)$ defined by $h(x) := f'(x_0)^{-1}f(x), x \in \Omega$, so that $h'(x) = f'(x_0)^{-1}f'(x) \in \mathcal{L}(X), x \in \Omega$, and $h'(x_0) = \mathrm{id}_X$. In terms of the function h, the assumptions of Theorem 5 therefore become

$$\|h(x_0)\| \le \frac{r}{2} \quad \text{and} \quad \|h'(\widetilde{x}) - h'(x)\| \le \frac{1}{r}\|\widetilde{x} - x\| \quad \text{for all } \widetilde{x}, x \in B(x_0;r).$$

(i) *The following estimates hold*:

$$\|h'(x)^{-1}\| \le \frac{1}{1 - \|x - x_0\|/r} \quad \text{for all } x \in B(x_0;r),$$

$$\|h(\widetilde{x}) - h(x) - h'(x)(\widetilde{x} - x)\| \le \frac{1}{2r}\|\widetilde{x} - x\|^2 \quad \text{for all } \widetilde{x},\ x \in \overline{B(x_0;r)}.$$

By assumption,

$$\|h'(x) - h'(x_0)\| = \|h'(x) - \mathrm{id}_X\|_{\mathcal{L}(X)} \le \frac{1}{r}\|x - x_0\| < 1 \quad \text{at each } x \in B(x_0;r).$$

Therefore, at each $x \in B(x_0;r)$, $h'(x)$ is one-to-one and onto, and (Sec. 2.1)

$$\begin{aligned}\|h'(x)^{-1}\| &\le \frac{\|h'(x_0)^{-1}\|}{1 - \|h'(x_0)^{-1}(h'(x) - h'(x_0))\|} \\ &= \frac{1}{1 - \|h'(x) - h'(x_0)\|} \le \frac{1}{1 - \|x - x_0\|/r}.\end{aligned}$$

Hence, the first estimate holds.

Using the mean value theorem (Theorem 1), we next have

$$\begin{aligned}\|h(\widetilde{x}) - h(x) - h'(x)(\widetilde{x} - x)\| &= \left\| \int_0^1 (h'((1-t)x + t\widetilde{x}) - h'(x))(\widetilde{x} - x)\,\mathrm{d}t \right\| \\ &\le \left(\int_0^1 \|h'((1-t)x + t\widetilde{x}) - h'(x)\|\,\mathrm{d}t \right) \|\widetilde{x} - x\| \\ &\le \frac{1}{r}\left(\int_0^1 t\,\mathrm{d}t \right) \|\widetilde{x} - x\|^2 \\ &= \frac{1}{2r}\|\widetilde{x} - x\|^2 \quad \text{for all } \widetilde{x}, x \in B(x_0;r).\end{aligned}$$

But the above inequality holds as well for all $\widetilde{x}, x \in \overline{B(x_0;r)}$ since the functions appearing on each side are continuous. Hence, the second estimate holds.

(ii) *The Newton iterates* $x_k, k \ge 0$, *for the function* h, *which are the same as those for the function* f, *belong to the open ball* $B(x_0;r)$ (hence, they are well defined) *and they satisfy the following estimates for all* $k \ge 1$:

$$\|x_k - x_{k-1}\| \le \frac{r}{2^k}, \quad \|x_k - x_0\| \le r\left(1 - \frac{1}{2^k}\right),$$

$$\|h'(x_k)^{-1}\| \le 2^k, \quad \|h(x_k)\| \le \frac{r}{2^{2k+1}}.$$

First, let us check that the above estimates hold for $k = 1$. Clearly, the point $x_1 = x_0 - h'(x_0)^{-1}h(x_0) = x_0 - h(x_0)$ is well defined since $h'(x_0)$ is invertible. Besides,

$$\|x_1 - x_0\| = \|h(x_0)\| \leq \frac{r}{2},$$

and, by (i),

$$\|(h'(x_1))^{-1}\| \leq \frac{1}{1 - \|x_1 - x_0\|/r} \leq 2.$$

By definition of x_1, and by (i) again,

$$\|h(x_1)\| = \|h(x_1) - h(x_0) - h'(x_0)(x_1 - x_0)\| \leq \frac{1}{2r}\|x_1 - x_0\|^2 = \frac{r}{2^3}.$$

So, assume that the estimates hold for $k = 1, \ldots, n$ for some integer $n \geq 1$. The point $x_{n+1} = x_n - h'(x_n)^{-1}h(x_n)$ is thus well defined since $h'(x_n)$ is invertible. Moreover, by the induction hypothesis and by the estimates of (i) (for the third and fourth estimates),

$$\|x_{n+1} - x_n\| \leq \|h'(x_n)^{-1}\|\,\|h(x_n)\| \leq \frac{r}{2^{n+1}},$$

$$\|x_{n+1} - x_0\| \leq \|x_n - x_0\| + \|x_{n+1} - x_n\| \leq r\left(1 - \frac{1}{2^n}\right) + \frac{r}{2^{n+1}} = r\left(1 - \frac{1}{2^{n+1}}\right),$$

$$\|h'(x_{n+1})^{-1}\| \leq \frac{1}{1 - \|x_{n+1} - x_0\|/r} \leq 2^{n+1},$$

$$\begin{aligned}\|h(x_{n+1})\| &= \|h(x_{n+1}) - h(x_n) - h'(x_n)(x_{n+1} - x_n)\| \\ &\leq \frac{1}{2r}\|x_{n+1} - x_n\|^2 \leq \frac{r}{2^{2(n+1)+1}}.\end{aligned}$$

Hence, the estimates also hold for $k = n + 1$.

(iii) *The Newton iterates $x_k, k \geq 0$, converge to a zero a of h, hence of f, which belongs to the closed ball $\overline{B(x_0; r)}$. Besides,*

$$\|x_k - a\| \leq \frac{r}{2^k} \quad \text{for all } k \geq 0.$$

The estimates $\|x_k - x_{k-1}\| \leq r/2^k, k \geq 1$, established in (ii) clearly imply that $(x_k)_{k=1}^{\infty}$ is a Cauchy sequence. Since $x_k \in B(x_0; r) \subset \overline{B(x_0; r)}$ and $\overline{B(x_0; r)}$ is a complete metric space (as a closed subset of the Banach space X), there exists $a \in \overline{B(x_0; r)}$ such that

$$a = \lim_{k \to \infty} x_k.$$

Since $\|h(x_k)\| \leq r/2^{2k+1}, k \geq 1$, by (ii), and h is a continuous function,

$$h(a) = \lim_{k \to \infty} h(x_k) = 0.$$

Hence, the point a is a zero of f.

Given integers $k \geq 1$ and $\ell \geq 1$, we have, again by (ii),

$$\|x_k - x_{k+\ell}\| \leq \sum_{j=k}^{\ell+p-1} \|x_{j+1} - x_j\| \leq \sum_{j=k}^{k+p-1} \frac{r}{2^{j+1}} < \sum_{j=k}^{\infty} \frac{r}{2^{j+1}} = \frac{r}{2^k},$$

so that, for each $k \geq 1$,

$$\|x_k - a\| = \lim_{\ell \to \infty} \|x_k - x_{k+\ell}\| \leq \frac{r}{2^k}.$$

(iv) *Uniqueness of a zero of h, hence of f, in the closed ball $\overline{B(x_0; r)}$.*

We first show that, if $b \in \overline{B(x_0; r)}$ is such that $h(b) = 0$, then

$$\|x_k - b\| \leq \frac{r}{2^k} \quad \text{for all } k \geq 0.$$

Clearly, this is true if $k = 0$; so, assume that this inequality holds for $k = 1, \ldots, n$, for some integer $n \geq 0$. Noting that we can write

$$x_{n+1} - b = x_n - h'(x_n)^{-1}h(x_n) - b = h'(x_n)^{-1}(h(b) - h(x_n) - h'(x_n)(b - x_n)),$$

we infer from (i) and (ii) and from the induction hypothesis that

$$\|x_{n+1} - b\| \leq \|h'(x_n)^{-1}\| \frac{1}{2r} \|b - x_n\|^2 \leq \frac{r}{2^{n+1}}.$$

Hence, the inequality $\|x_k - b\| \leq r/2^k$ holds for all $k \geq 1$. Consequently,

$$\lim_{n \to \infty} \|x_k - b\| = \|a - b\| = 0,$$

which shows that $b = a$. This completes the proof. □

References

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (Springer, Berlin, 2010).

[2] J. P. Dedieu, *Points Fixes, Zéros et la Méthode de Newton* (Springer, Berlin, 2006).

[3] K. Deimling, *Nonlinear Functional Analysis* (Springer-Verlag, 1985).

[4] P. Deuflhard, *Newton Methods for Nonlinear Problems — Affine Invariance and Adaptive Algorithms* (Springer, Berlin, 2004).

[5] J. Dieudonné, *Foundations of Modern Analysis* (Academic Press, New York, 1960).

[6] W. B. Gragg and R. A. Tapia, Optimal error bounds for the Newton–Kantorovich theorem, *SIAM J. Numer. Anal.* **11** (1974) 10–13.

[7] L. V. Kantorovich, Functional analysis and applied mathematics, *Uspekhi Math. Nauk* **3** (1948) 89–185.

[8] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces* (Fizmatgiz, Moscow, 1959); English translation (Macmillan, New York, 1964).

[9] S. Lang, *Real and Functional Analysis*, 3rd edn. (Springer-Verlag, New York, 1993).

[10] J. M. Ortega, The Newton–Kantorovich theorem, *Amer. Math. Monthly* **75** (1968) 658–660.

[11] W. C. Rheinboldt, A unified convergence theory for a class of iterative processes, *SIAM J. Numer. Anal.* **5** (1968) 42–63.

[12] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd edn. (John Wiley, 1980).
[13] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, *Volume I*: *Fixed-Point Theorems* (Springer-Verlag, Berlin, 1986).
[14] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, *Volume II/B*: *Nonlinear Monotone Operators* (Springer-Verlag, Berlin, 1990).