MATH 5510 – Numerical Analysis October 24, 2016

# Assignment 6

1. Describe the solution to the least square problem

$$\min_x \|Ax - b\|_2^2,$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, but using the QR-decomposition with a permutation, i.e.

$$AP = QR,$$

where $Q$ is orthogonal, $R$ upper triangular and $P$ is a permutation matrices.

2. Suppose we want to fit the data $(t_i, b_i)$, $i = 1, \ldots, m$ by using

$$b(t) = x_1\varphi_1(t) + x_2\varphi_2(t) + \cdots + x_n\varphi_n(t).$$

We are looking for $x_1, x_2, \ldots, x_n$ that solve

$$\min_x \sum_{i=1}^{m} \omega_i \left(b_i - x_1\varphi_1(t_i) - x_2\varphi_2(t_i) - \cdots - x_n\varphi_n(t_i)\right)^2,$$

where $\omega_i > 0$, $i = 1, \ldots, m$, are some weights. Formulate the above problem as a linear least square problem,

$$\min_x \|\tilde{A}x - \tilde{b}\|_2^2.$$

3. Show that the linear least square problem

$$\min_x \|Ax - b\|_2$$

is equivalent to the system

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

4. Show that a matrix $A = uv^T$ for $u \in \mathbb{R}^n$ and $v \in R^m$ has rank 1.

5. Let $A = U\Sigma V^T$ be the svd decomposition of $A \in \mathbb{R}^{m\times n}$, with diagonal entries of $\Sigma$,

$$\sigma_1^2 \geq \sigma_2^2 \cdots \geq \sigma_r^2 > \sigma_{r+1}^2 = \cdots = \sigma_{\min\{m,n\}}^2 = 0.$$

For $k \leq r$ define

$$x_k = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i^2} v_i.$$

Show that

$$\|x_{k-1}\|_2 \leq \|x_k\|_2$$

and

$$\|Ax_{k-1} - b\|_2 \geq \|Ax_k - b\|_2.$$