MATH 5510 – Numerical Analysis 1. November 30, 2016

Final Project 3

Consider a nonlinear two-point boundary value problem

$$\begin{aligned} -u''(x) + u^p(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{1}$$

where $p \geq 2$ and $f(x)$ is a given continuous function on $[0, 1]$. Show that the solution $u(x)$ for $0 < x < 1$ to (1) can be expressed as

$$u(x) = \int_0^1 G(x,t)(f(t) - u^p(t))dt$$

where

$$G(x,t) = \begin{cases} t(1-x), & 0 \leq t \leq x \leq 1 \\ x(1-t), & 0 \leq x \leq t \leq 1. \end{cases}$$

Solving this integral equation in turn amounts to finding a zero of the nonlinear mapping

$$g : u \to g(u)$$

defined by

$$g(u(x)) = u(x) + \int_0^1 G(x,t)(u^p(t) - f(t))dt, \quad 0 \leq x \leq 1.$$

In the paper *On the Newton-Kantorovich Theorem* by P.G.Ciarlet and C.Mardare it is shown that the assumptions of the Newton Kantorovich theorem are satisfied. Thus (1) has a solution and this solution can be approximated by Newtons method. Given the k-th Newton iterate u_k, finding the $(k+1)$-th iterate u_{k+1} amounts to solving the linear boundary value problem

$$\begin{aligned} -u''(x) + pu_k^{p-1}(x)u(x) &= (p-1)u_k^p(x) - f(x), \quad 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{2}$$

Write a Matlab function

```
[U] = my_FD_two_point(N,f,Tol)
```

that solves the following two point boundary value problem (1) for $p = 3$ by using the above iterative Newton's method with finite difference scheme given in class. To initialize Newton you may take $u_0 = 0$. You need to develop a termination strategy for the Newton's method. The input for the function is the following:

- $N + 1$– number of the mesh points in the method
- $f(x)$ – inline function for the RHS
- Tol – a tolerance, a small positive number

The output should be the $(N + 1) \times 1$ solution vector U.

To check your program run the following script (can be downloaded from course website). E-mail me your function. Print and submit the script of your function, the tables and the figures produced from running the test script. Make sure that the names of your functions match.

```
% Final_project3_check

N = 20; % number of mesh points
f = inline('sin(pi*x)*pi^2+sin(pi*x).^3,'x'); % RHS
Tol = 1e-10;

% computing solution FD method
U = my_FD_two_point(N,f,Tol);

x = linspace(0,1,N+1);
% exact solution
U_ex = sin(pi*x);
% plotting exact and the approximate solution
figure(1)
plot(x,U(:),'r*', x,U_ex);


% check space rate of convergence
fprintf(['   dx     error        rate of convergence         \n'] )
   disp('-----------------------------------------------------------------')

error = zeros(10,1);
N = 10;
U = my_FD_two_point(N,f,Tol);
x = linspace(0,1,N+1)';
U_ex = sin(pi*x);
error(1) = norm(U(:)-U_ex(:),inf);
for j = 2:5
       N = 10*2^(j-1);
       U = my_FD_two_point(N,f,Tol);
       x = linspace(0,1,N+1)';
       U_ex = sin(pi*x);
       error(j) = norm(U(:)-U_ex(:),inf);
       dx = 2/(10*2^(j-1));
       fprintf([' %1.2e  %20.3e    %20.5f   \n'],...
       dx, error(j), log2(error(j-1)/error(j))  )
end
disp('-----------------------------------------------------------------')
```