Assignment 3

1. Let $x_1, \ldots, x_n \in [a, b]$, $w_1, \ldots, w_n \in \mathbb{R}$ be the nodes and the weight of a quadrature formula. Assume that $w_j < 0$ for some $j \in \{1, \ldots, n\}$. Construct a continuous function $f : [a, b] \to \mathbb{R}$ such that $f(x) \geq 0$, $x \in [a, b]$, i.e.,
$$\int_a^b f(x)dx > 0,$$
but
$$\sum_{i=1}^{n} w_i f(x_i) < 0.$$

2. Write a program that approximates
$$\int_a^b f(x)dx$$
by the composite trapezoidal rule $T(h)$. Starting with $h = (b - a)$ compute $|T(h) - T(h/2)|/|T(h/2)|$ and reduce h by a factor of 2 until $|T(h) - T(h/2)|/|T(h/2)| < 10^{-6}$. Your program should return a table with $h/2$, $T(h/2)$, $|T(h) - T(h/2)|/|T(h/2)|$ and it should return the total number of function evaluations $f(x)$. Your program should reuse computed function values as much as possible.

   Apply your program to approximate the integrals
$$\int_0^3 \frac{x}{1+x^2}dx = \frac{1}{2}\ln 10,$$
$$\int_0^{0.95} \frac{1}{1-x}dx = \ln 20,$$
$$\int_0^{\pi/2} \frac{1}{\sqrt{1 - m\sin^2 x}}dx, \quad m = 0.5, 0.8, 0.95.$$

3. Show that the composite trapezoidal rule
$$\int_0^{2\pi} f(x)dx \approx \frac{2\pi}{n}\sum_{j=0}^{n-1} f\left(\frac{2\pi j}{n}\right)$$
is exact for all trigonometric polynomials of degree less or equal to $n - 1$, i.e. for all trigonometric polynomials of the form
$$\sum_{k=0}^{n-1} \left(a_k \cos(kx) + b_k \sin(kx)\right).$$
(Hint: Notice that $\sum_{k=0}^{n-1} \left(a_k \cos(kx) + b_k \sin(kx)\right) = \sum_{k=-(n-1)}^{n-1} c_k e^{ikx}$ for some c_k.)

4. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ cannot be evaluated exactly, but instead of $f(x)$ a value $\hat{f}(x) = f(x) + \delta(x)$ is computed, where $|\delta(x)| \leq \delta$ for all $x \in \mathbb{R}$. Suppose one is interested in the integral
$$\int_a^b f(x)dx$$
and wants to approximate the integral by the composite trapezoidal rule $T(h)$ with step size $h = (b - a)/N$.

(a) Derive an estimate for the error

$$|\int_a^b f(x)dx - T(h)|$$

when inexact function evaluations are used.

(b) Given δ, what would the range of "reasonable" step sizes h be? Justify your answer. Demonstrate your findings by approximating

$$\int_0^1 x^3 dx,$$

where instead of

$$f(x) = x^3$$

the function

$$\hat{f}(x) = x^3 + 0.01 * \texttt{rand}$$

is used in the composite trapezoidal formula.