MATH 5510 – Numerical Analysis September 13, 2013

Assignment 2

1. Define Tchebyshev polynomials: $T_n(x) = \cos(n \cos^{-1} x);\ n = 0, 1, \ldots$.

   (a) Derive the recurrence relation

   $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

   (b) Show that $T_n(x)$ has n zeros in the interval $(-1, 1)$ that are given by

   $$x_k = \cos\left(\frac{2k+1}{2n}\pi\right), \quad k = 0, 1, \ldots, n-1.$$

   (c) Prove that $\hat{T}_n(x) = \frac{T_n(x)}{2^{n-1}}$ is monic (i.e. the leading coefficient is 1).

   (d) Find the values x_1 and x_2 for which

   $$\max_{x\in[-1,1]} |(x - x_1)(x - x_2)|$$

   is minimized. What is the connection between $T_2(x)$ and $(x - x_1)(x - x_2)$?

2. A function f is defined on the interval $[0, 1]$ and its derivatives satisfy that $|f^{(m)}(x)| \le m!$, for all $x \in [0, 1]$ and for all $m = 0, 1, \ldots$. For any $0 < q < 1$, let $p_n(x)$, $n > 0$, be the interpolation polynomial of degree at most n which interpolates f at $x_0 = 1$, $x_1 = q$, $x_2 = q^2$, $\ldots$, $x_n = q^n$. Show that

   $$\lim_{n\to\infty} p_n(0) = f(0)$$

   Taking $q = 1/2$ and $n = 10$, find an upper estimate on $|p_{10}(0) - f(0)|$.

3. Similarly to the class presentation for the natural boundary close the system for the cubic splines for

   (a) Clamped boundary $S'(x_1) = f'(x_1)$, $S'(x_{n+1}) = f'(x_{n+1})$.

   (b) Periodic boundary $S(x_1) = S(x_{n+1})$, $S'(x_1) = S'(x_{n+1})$, $S''(x_1) = S''(x_{n+1})$.