

MATH 3511

Lecture 2. Matrix Algebra

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Goals

- ▶ Matrix-matrix multiplication.
- ▶ Inverse of a Matrix.
- ▶ Determinant.

Properties of Matrices

A space of all matrices with m rows and n columns

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

we will denote by $\mathbb{R}^{m \times n}$

Usually we denote matrices capital letters and vectors we denote by the small ones.

Properties of Matrices

A useful way to look at matrix $A \in \mathbb{R}^{m \times n}$ as

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix},$$

i.e. a collection of n vectors in $\mathbb{R}^m$.

Properties of Matrices

Definition

Two matrices A and B are equal if they have the same number of rows and columns, i.e. $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$ and if $a_{ij} = b_{ij}$, for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Definition

If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$ then $A + B = C \in \mathbb{R}^{m \times n}$ with entries $c_{ij} = a_{ij} + b_{ij}$, for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Definition

If $A \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{R}$ then $\lambda A \in \mathbb{R}^{m \times n}$ with entries λa_{ij} , for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Properties of Matrices

Example

Let

$$A = \begin{pmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{pmatrix}, \quad \lambda = 2,$$

then

$$A + B = \begin{pmatrix} 6 & 1 & -1 \\ 3 & 2 & 6 \end{pmatrix}$$

and

$$\lambda A = \begin{pmatrix} 4 & -2 & 14 \\ -6 & -2 & 0 \end{pmatrix}$$

Matrices

Let $A, B, C, O \in \mathbb{R}^{m \times n}$, where O is a matrix with all entries to be zero, and let $\lambda, \mu \in \mathbb{R}$. The following properties is easy to show

- ▶ $A + B = B + A$
- ▶ $(A + B) + C = A + (B + C)$
- ▶ $A + O = O + A = A$
- ▶ $A + (-A) = A - A = O$
- ▶ $\lambda(A + B) = \lambda A + \lambda B$
- ▶ $(\lambda + \mu)A = \lambda A + \mu A$
- ▶ $\lambda(\mu A) = (\lambda\mu)A$

Matrices

Definition

A matrix is called *square* if it has the same number of rows as columns.

Definition

A matrix $D = \{d_{ij}\}$ is called *diagonal* if it is square and $d_{ij} = 0$ whenever $i \neq j$.

Definition

A matrix I is called *an identity matrix of order* if it is diagonal and all diagonal entries are equal to 1.

Matrix-Vector Multiplication

Let $A \in R^{m \times n}$ and $x \in R^n$. The i -th component of the matrix-vector product $y = Ax$ is defined by

$$y_i = \sum_{j=1}^n a_{ij}x_j \quad (1)$$

i.e., y_i is the dot product (inner product) of the i -th row of A with the vector x .

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$$\begin{pmatrix} \vdots \\ \hline y_i \\ \hline \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \hline a_{i1} & a_{i2} & a_{in} \\ \hline \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Matrix-Vector Multiplication

Another useful point of view is to look at entire vector $y = Ax$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Thus, y is a linear combination of the columns of matrix A .

Matrix-Matrix Multiplication

If $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$, then $AB = C \in \mathbb{R}^{m \times n}$ defined by

$$c_{ij} = \sum_{k=1}^p a_{ik} b_{kj},$$

i.e. the ij -th element of the product matrix is the dot product between i -th row of A and j -th column of B .

$$\begin{pmatrix} c_{ij} \end{pmatrix} = \begin{pmatrix} \overline{a_{i1} \quad \cdots \quad a_{ip}} \end{pmatrix} \begin{pmatrix} \left| \begin{array}{c} b_{1j} \\ \vdots \\ b_{pj} \end{array} \right| \end{pmatrix}$$

Thus, y is a linear combination of the columns of matrix A .

Properties of Matrices

Example

Let

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 4 & 1 \\ 10 & 15 \end{pmatrix}.$$

Matrix-Matrix Multiplication

Another useful point of view is to look at j -th column of C

$$\begin{pmatrix} c_{1j} \\ \vdots \\ c_{ij} \\ \vdots \\ c_{mj} \end{pmatrix} = b_{1j} \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + b_{2j} \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + b_{nj} \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Thus, j -th column of C is a linear combination of the columns of matrix A .

Matrix-Matrix Multiplication (cont.)

Sometimes it is useful to consider matrices partitioned into blocks. For example,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with $m_1 + m_2 = m$, $p_1 + p_2 = p$, and $n_1 + n_2 = n$.

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with $m_1 + m_2 = m$, $p_1 + p_2 = p$, and $n_1 + n_2 = n$.

This time $C = AB$ can be expressed as

$$C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{12}B_{12} + A_{22}B_{22} \end{pmatrix}.$$

Matrix-Matrix Multiplication (cont.)

For example, if

$$A = \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array} \right) \quad B = \left(\begin{array}{c|c} 1 & 2 \\ 3 & 4 \\ \hline 5 & 6 \end{array} \right),$$

then

Matrix-Matrix Multiplication (cont.)

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This time $C = AB$ can be expressed as

$$C = \left(\begin{array}{c|c} \left(\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} 5 \right) & \left(\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} 6 \right) \\ \hline \left(\begin{pmatrix} 7 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 9 \cdot 5 \right) & \left(\begin{pmatrix} 7 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 9 \cdot 6 \right) \end{array} \right)$$
$$= \left(\begin{array}{c|c} 22 & 28 \\ 49 & 64 \\ \hline 76 & 100 \end{array} \right).$$

Properties of Matrices (cont.)

- ▶ Matrix multiplication even for square matrices in general is not commutative, i.e. usually $AB \neq BA$.
- ▶ Straight implementation of matrix-matrix multiplication requires $O(n^3)$ operations. Using block partition with smart manipulation one can reduce the asymptotic number of operations. See original paper

Strassen and Volker, Gaussian Elimination is not Optimal

Basic Linear Algebra Subroutines (BLAS)

- ▶ There are many mathematically equivalent ways to organize basic linear algebra tasks, such as matrix-vector multiplications, matrix-matrix-multiplications, etc.

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- ▶ Write code in terms of Basic Linear Algebra Subroutines (BLAS)
BLAS level 1: vector operations such as $x + y$.
BLAS level 2: matrix vector operations such as Ax .
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BLAS level 1: vector operations such as $x + y$.
BLAS level 2: matrix vector operations such as Ax .
BLAS level 3: matrix matrix operations such as AB .
- ▶ Use Automatically Tuned Linear Algebra Software (ATLAS)
<http://math-atlas.sourceforge.net/> or other tuned (to the computer system of interest) BLAS.

Inverse Matrices

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is called **nonsingular** or **invertible** if there exists a matrix B such that $BA = AB = I$. If such matrix B exists it is called the **inverse** of A and is denoted by A^{-1} .

Definition

A matrix without an inverse is called **singular** or **noninvertible**.

Inverse Matrices

Let $A \in \mathbb{R}^{n \times n}$ be any nonsingular matrix. One can show the following:

- ▶ A^{-1} is unique.
- ▶ A^{-1} is again nonsingular matrix and $(A^{-1})^{-1} = A$.
- ▶ If $B \in \mathbb{R}^{n \times n}$ is another nonsingular matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Transpose of a Matrix

► Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Transpose of a Matrix

- ▶ Let

$$A = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & & & \\ 4 & 5 & 6 & & & \\ 7 & 8 & 9 & & & \end{array} \right) \quad \text{then} \quad A^T = \left(\begin{array}{ccc|ccc} 1 & 4 & 7 & & & \\ 2 & 5 & 8 & & & \\ 3 & 6 & 9 & & & \end{array} \right).$$

- ▶ If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$, then

$$(AB)^T = B^T A^T.$$

More generally,

$$(A_1 A_2 \dots A_j)^T = A_j^T \dots A_2^T A_1^T.$$

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More generally,

$$(A_1 A_2 \dots A_j)^T = A_j^T \dots A_2^T A_1^T.$$

- ▶ If $A \in \mathbb{R}^{n \times n}$ is invertible, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T.$$

We will write A^{-T} .

Determinant of a matrix

Let $A \in \mathbb{R}^{n \times n}$ be any square matrix. The determinant of a matrix provides existence and uniqueness results for linear systems having the same number of equations and unknowns. We will denote the determinant of a matrix A by $\det A$. Another common notation is $|A|$.

Determinant of a matrix (cont.)

Definition

- ▶ If $A = [a]$ is a 1-by-1 matrix, then $\det A = a$.
- ▶ If A is an n -by- n matrix, with $n > 1$ the minor M_{ij} is the determinant of the $(n-1)$ -by- $(n-1)$ submatrix of A obtained by deleting the i -th row and j -th column of the matrix A .
- ▶ The cofactor A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.
- ▶ The determinant of the n -by- n matrix A , when $n > 1$, is given either by

$$\det A = \sum_{j=1}^n a_{ij} A_{ij} = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \dots, n,$$

or by

$$\det A = \sum_{i=1}^n a_{ij} A_{ij} = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } j = 1, 2, \dots, n.$$

Determinant of a matrix (cont.)

Theorem

Let $A \in \mathbb{R}^{n \times n}$

1. $\det A^T = \det A$.
2. If any row or column of A has only zero entries, then $\det A = 0$.
3. If A has two rows or two columns the same, then $\det A = 0$.
4. If $\tilde{A}$ is obtained from A by interchanging two rows or columns then $\det \tilde{A} = -\det A$.
5. If $\tilde{A}$ is obtained from A by multiplying any row or column by λ , then $\det \tilde{A} = \lambda \det A$.
6. If $\tilde{A}$ is obtained from A by replacing row (column) $A(i, :)$ by $A(i, :) + \lambda A(j, :)$ with $i \neq j$, then $\det \tilde{A} = \det A$.
7. If B is any n -by- n matrix, then $\det AB = \det A \det B$.
8. If A^{-1} exists, $\det A^{-1} = (\det A)^{-1}$.
9. If A is an upper or lower triangular, then $\det A = \prod_{i=1}^n a_{ii}$.

Determinant of a matrix (cont.)

Theorem

The following statements are equivalent for any n -by- n matrix A :

- 1. The equation $Ax = 0$ has a unique solution $x = 0$.*
- 2. The system $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.*
- 3. The matrix A is nonsingular; that is, A^{-1} exists.*
- 4. $\det A \neq 0$.*
- 5. Gaussian elimination produces a matrix with non-zero pivots.*