

MATH 3511

Lecture 5. Solving Linear Systems 2

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Goals

- ▶ Understand what happens when you type $[L, U, P] = lu(A)$.
- ▶ Understand how to use matrix decompositions.
- ▶ Computational aspects.

Motivation for LU Decomposition

- ▶ The present form of the Gaussian elimination with partial pivoting is useful to solve a linear system $Ax = b$.
- ▶ However, we need it to be more versatile. For example, suppose we have used Gaussian elimination with partial pivoting to solve $Ax = b$ (cost $\approx 2n^3/3$ flops, where n is the size of the system). If we are given another right hand side c and are asked to solve $Ax = c$, currently we have to redo all computations and solve the new linear system at a cost of $\approx 2n^3/3$ flops, even though most of the operations are identical to the ones we have executed before. (This situation arises, for example, in the numerical solution of time dependent differential equations.)

Motivation for LU Decomposition

- ▶ The present form of the Gaussian elimination with partial pivoting is useful to solve a linear system $Ax = b$.
- ▶ However, we need it to be more versatile. For example, suppose we have used Gaussian elimination with partial pivoting to solve $Ax = b$ (cost $\approx 2n^3/3$ flops, where n is the size of the system). If we are given another right hand side c and are asked to solve $Ax = c$, currently we have to redo all computations and solve the new linear system at a cost of $\approx 2n^3/3$ flops, even though most of the operations are identical to the ones we have executed before. (This situation arises, for example, in the numerical solution of time dependent differential equations.)
- ▶ The LU-decomposition removes these shortcomings.
- ▶ We will first reorganize the computations performed in the Gaussian elimination with partial pivoting and separate computations performed on A from those performed on the right hand side. Then we will interpret these operations mathematically to be able to reuse the computations to perform other tasks, for example to solve the linear system $A^T y = c$.

Gaussian Elimination Revisited

- ▶ Gaussian elimination operates on the augmented matrix $(A \mid b)$ and performs two types of operations to transform $(A \mid b)$ into $(U \mid y)$, where U is upper triangular.
- ▶ The right hand side does not influence how the augmented matrix $(A \mid b)$ is transformed into an upper triangular matrix $(U \mid y)$. This transformation depends only on the matrix A .

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- ▶ What operations are performed? In step k of the Gaussian elimination with partial pivoting we perform two operations:
 - ▶ Interchange a row $i_0 > k$ with row k .
 - ▶ Add a multiple $-l_{ik}$ times row k to row i for $i = k + 1, \dots, n$.

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- ▶ What operations are performed? In step k of the Gaussian elimination with partial pivoting we perform two operations:
 - ▶ Interchange a row $i_0 > k$ with row k .
 - ▶ Add a multiple $-l_{ik}$ times row k to row i for $i = k + 1, \dots, n$.
- ▶ How do we record this?
 - ▶ Introduce an integer array *ipivt*. Set

$$ipivt(k) = i_0$$

when the k -th step the rows k and i_0 are interchanged.

- ▶ Store the $-l_{ik}$ in the (i, k) -th position of the array that originally held A . (Remember that we add a multiple $-l_{ik}$ times row k to row i to eliminate the entry (i, k) . Hence this storage can be reused.)

Example

Consider the matrix

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$$ipivt = (2, \ , \) \quad \left(\begin{array}{c|cc} 2 & 1 & 0 \\ \hline -\frac{1}{2} & \frac{3}{2} & -1 \\ \frac{1}{2} & \frac{5}{2} & 2 \end{array} \right)$$

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After step $k = 2$ the vector $ipivt$ and the matrix A are given by

$$ipivt = (2, 3, \) \quad \left(\begin{array}{c|cc} 2 & 1 & 0 \\ -\frac{1}{2} & \frac{5}{2} & 2 \\ \frac{1}{2} & -\frac{3}{5} & -\frac{11}{5} \end{array} \right)$$

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$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{step 1}} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \xrightarrow{\text{step 2}} \begin{pmatrix} 2 \\ 2 \\ -11/5 \end{pmatrix}$$

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and then solve

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & \frac{5}{2} & 2 \\ 0 & 0 & -\frac{11}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -11/5 \end{pmatrix}$$

by back substitution to obtain $x_1 = 1$, $x_2 = 0$, $x_3 = 1$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$



Step 1



$$\left(\begin{array}{c|cccccc} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1,n-1} & a_{1n} \\ -l_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2,n-1} & a_{2n} \\ -l_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ -l_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ -l_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{array} \right)$$



Step 2



Step 2



$$\left(\begin{array}{c|c|c|cc|c|c} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1,n-1} & a_{1n} \\ -l_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2,n-1} & a_{2n} \\ -l_{31} & -l_{32} & a_{33} & \cdots & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ -l_{n-1,1} & -l_{n-1,2} & a_{n-1,3} & \cdots & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ -l_{n1} & -l_{n2} & a_{n3} & \cdots & \cdots & a_{n,n-1} & a_{nn} \end{array} \right)$$



Step n-1



$$\left(\begin{array}{c|c|c|cc|c|c} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1,n-1} & a_{1n} \\ -l_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2,n-1} & a_{2n} \\ -l_{31} & -l_{32} & a_{33} & \cdots & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & a_{n-1,n-1} & a_{n-1,n} \\ -l_{n1} & -l_{n2} & -l_{n3} & \cdots & \cdots & -l_{n,n-1} & a_{nn} \end{array} \right)$$

Gaussian Elimination via Matrix Multiplications

Row interchange in step k can be expressed by multiplying with

$$P_k = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & 1 & & & & & 0 & \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \begin{matrix} \leftarrow k \\ \\ \\ \leftarrow \text{ipivt}(k) \\ \\ \end{matrix}$$

from the left. P_k is a permutation matrix

Gaussian Elimination via Matrix Multiplications (cont)

Adding $-l_{i,k}$ times row k to row i for $i = k + 1, \dots, n$ is equivalent to multiplying from the left with

$$M_k = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & & \\ & & \vdots & \ddots & \\ & & -l_{n,k} & & 1 \end{pmatrix}$$

Observe that matrix M_k is lower triangular and is called a Gauss transformation.

Gaussian Elimination via Matrix Multiplications (cont)

- ▶ The transformation of a matrix A into an upper triangular matrix U can be expressed as a sequence of matrix-matrix multiplications

$$M_{n-1}P_{n-1} \dots M_1P_1A = U.$$

Gaussian Elimination via Matrix Multiplications (cont)

- ▶ The transformation of a matrix A into an upper triangular matrix U can be expressed as a sequence of matrix-matrix multiplications

$$M_{n-1}P_{n-1} \dots M_1P_1A = U.$$

- ▶ We will show that P_k and M_k are invertible. Hence, if we have to solve $Ax = b$ and if

$$M_{n-1}P_{n-1} \dots M_1P_1A = U,$$

then we apply the matrices to the right hand side,

$$M_{n-1}P_{n-1} \dots M_1P_1b = y$$

and solve

$$Ux = y.$$

Gaussian Elimination via Matrix Multiplications (cont)

Example

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}$$

Step 1:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}, \quad M_1 P_1 A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 5/2 & 2 \end{pmatrix}$$

Gaussian Elimination via Matrix Multiplications (cont)

Example

Step 2:

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 M_1 P_1 A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5/2 & 2 \\ -1 & 3/2 & -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/5 & 1 \end{pmatrix}, \quad M_2 P_2 M_1 P_1 A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5/2 & 2 \\ 0 & 0 & -11/5 \end{pmatrix}$$

Gaussian Elimination via Matrix Multiplications (cont)

Example

For $b = (0, 2, 1)^T$ solve $Ax = b$.

Compute

$$\begin{aligned} M_2 P_2 M_1 P_1 b &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \\ -\frac{11}{5} \end{pmatrix} \end{aligned}$$

Solving

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5/2 & 2 \\ 0 & 0 & -11/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -\frac{11}{5} \end{pmatrix}$$

we obtain $x = (1, 0, 1)^T$.

Properties of P_k

$$P_k = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & 1 & & & & & 0 & \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow k \\ \\ \\ \\ \leftarrow \text{ipivt}(k) \\ \\ \\ \end{matrix}$$

satisfies $P_k = P_k^T$ and $P_k^2 = Id \Rightarrow P_k^{-1} = P_k$.

$P_k A$ interchanges rows k and $\text{ipivt}(k)$ of A .

$A P_k$ interchanges columns k and $\text{ipivt}(k)$ of A .

Properties of M_k

If

$$M_k = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & & \\ & & \vdots & \ddots & \\ & & -l_{n,k} & & 1 \end{pmatrix}$$

then

$$M_k^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & l_{k+1,k} & & \\ & & \vdots & \ddots & \\ & & l_{n,k} & & 1 \end{pmatrix}$$

$M_k M_j$ is lower triangular.

Gaussian Elimination via Matrix Multiplications (cont.)

- We have

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Gaussian Elimination via Matrix Multiplications (cont.)

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- ▶ In practice the matrices $P_1, M_1 \dots$ are never computed explicitly. All information needed to represent these matrices is contained in the integer array *ipivt* and in the strictly lower triangular part of the array A , which at the end of the Gaussian Elimination holds the $-l_{ik}$'s for $k = 1, \dots, n-1$ and $i = k+1, \dots, n$.

Gaussian Elimination via Matrix Multiplications (cont.)

- We can perform operations with P_k and M_k as follows. Let $v = (v_1, \dots, v_n)^T$. Then if $ipivt(k) = i_0$

$$P_k v = (v_1, \dots, v_{k-1}, v_{i_0}, v_{k+1}, \dots, v_{i_0-1}, v_k, v_{i_0+1}, \dots, v_n)^T$$

(interchange entries k and $ipivt(k) = i_0$).

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- ▶ and $M_k v$ is

$$M_k v = (v_1, \dots, v_k, v_{k+1} - l_{k+1,k} v_k, \dots, v_n - l_{n,k} v_k)^T$$

(adds $-l_{i,k} v_k$ to v_i for $i = k + 1, \dots, n$).

LU Decomposition

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- ▶ We can show that for $j > k$

$$P_j M_k = \tilde{M}_k P_j, \tag{1}$$

where $\tilde{M}_k$ is obtained from M_k by interchanging the entries $-l_{j,k}$ and $-l_{\text{pivot}(j),k}$. $\tilde{M}_k$ has the same structure as M_k and we can easily compute $\tilde{M}_k^{-1}$.

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- ▶ Using (1) we can move all P_j 's to the right of the $\tilde{M}_k$'s

$$U = M_{n-1}P_{n-1} \dots M_1P_1A = M_{n-1}\tilde{M}_{n-2} \dots \tilde{M}_1P_{n-1} \dots M_1P_1A.$$

LU Decomposition (cont.)

► Now

$$U = M_{n-1}\tilde{M}_{n-2}\dots\tilde{M}_1P_{n-1}\dots M_1P_1A,$$

implies

$$P_{n-1}\dots M_1P_1A = \tilde{M}_1^{-1}\dots\tilde{M}_{n-2}^{-1}M_{n-1}^{-1}U$$

Thus $P = P_{n-1}\dots P_1$ is the permutation matrix and $L = \tilde{M}_1^{-1}\dots\tilde{M}_{n-2}^{-1}M_{n-1}^{-1}$ is a lower triangular matrix with diagonal elements equal to 1.

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Thus $P = P_{n-1}\dots P_1$ is the permutation matrix and $L = \tilde{M}_1^{-1}\dots\tilde{M}_{n-2}^{-1}M_{n-1}^{-1}$ is a lower triangular matrix with diagonal elements equal to 1.

- Matlab 's $[L,U,P] = lu(A)$ computes P, L, U such that $PA = LU$.

LU Decomposition (cont.)

Let

$$U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & 2 \\ 0 & 0 & -\frac{11}{5} \end{pmatrix} = M_2 P_2 M_1 P_1 A.$$

as before. Then using

$$\begin{aligned} P_2 M_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \tilde{M}_1 P_2 \end{aligned}$$

we have

LU Decomposition (cont.)

$$U = M_2 \tilde{M}_1 P_2 P_1 A.$$

Calling

$$L = \tilde{M}_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{3}{5} & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

we have $PA = LU$, i.e.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{3}{5} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & 2 \\ 0 & 0 & -\frac{11}{5} \end{pmatrix}.$$