

MATH 3511

Lecture 1. Solving Linear Systems 1

Dmitriy Leykekhman

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Goals

- ▶ Review of basic linear algebra
- ▶ Solution of simple linear systems.
- ▶ Gaussian elimination.

Linear System of Equations

Linear system of m equations with n unknowns $x_1, x_2, \dots, x_n$ we write as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m, \end{array}$$

where $a_{ij} \in \mathbb{R}$ and $b_i \in \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Linear System of Equations

From the coefficients a_{ij} we form a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We also define vectors b and x

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Notice that $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

Matrix-Vector Multiplication

Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$. The i -th component of the matrix-vector product $y = Ax$ is defined by

$$y_i = \sum_{j=1}^n a_{ij}x_j \quad (1)$$

i.e., y_i is the dot product (inner product) of the i -th row of A with the vector x .

$$\begin{pmatrix} \vdots \\ \hline y_i \\ \hline \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \hline a_{i1} & a_{i2} & a_{in} \\ \hline \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Matrix-Vector Multiplication

Another useful point of view is to look at entire vector $y = Ax$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Thus, y is a linear combination of the columns of matrix A .

Definitions

- ▶ Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be some vectors in $\mathbb{R}^n$ and $c_1, \dots, c_n$ be some numbers. An expression of the form

$$c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$$

is called a *linear combination* of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

- ▶ A set of all possible linear combinations of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is called a *linear span* of $\mathbf{v}_1, \dots, \mathbf{v}_n$.
- ▶ Vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are *linearly dependent* if there exist numbers $c_1, \dots, c_n$ not all zeros such that

$$c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = \mathbf{0}.$$

If such numbers do not exist we say that vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are *linearly independent*.

Solution of Linear Systems

- ▶ Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$. Then Ax is a linear combination of the columns of A .
Hence $Ax = b$ has a solution if $b \in \mathbb{R}^m$ can be written as a linear combination of the columns of A .
- ▶ **Solvability:**
 $Ax = b$ is solvable for every $b \in \mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$ (necessary $n \geq m$).
- ▶ **Uniqueness:**
If $Ax = b$ has a solution, then the solution is unique iff the columns of A are linearly independent (necessary $n \leq m$).
- ▶ Hence, for any $b \in \mathbb{R}^m$, the system $Ax = b$ has a unique solution iff $n = m$ and the columns of A are linearly independent.

Solution of Triangular Systems

- ▶ A matrix $L \in \mathbb{R}^{n \times n}$ is called *lower triangular matrix* if all matrix entries above the diagonal are equal to zero, i.e., if

$$l_{ij} = 0 \quad \text{for } j > i.$$

- ▶ A matrix $U \in \mathbb{R}^{n \times n}$ is called *upper triangular matrix* if all matrix entries below the diagonal are equal to zero, i.e., if

$$l_{ij} = 0 \quad \text{for } i > j.$$

- ▶ A Linear system with lower (upper) triangular matrix can be solved by forward substitution (backward substitution).

Solution of Triangular Systems

Example

Consider the system

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 4 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$

Forward substitution gives

$$x_1 = 4/2 = 2, \quad x_2 = (2 - 1 \cdot 2)/4 = 0, \quad x_3 = (5 - 4 \cdot 2 - 3 \cdot 0)/3 = -1.$$

Solution of Triangular Systems

Example

Consider the system

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

Backward substitution gives

$$x_3 = 6/3 = 2, \quad x_2 = (2 - 4 \cdot 2)/(-2) = 3, \quad x_1 = (4 - 2 \cdot 2 - 4 \cdot 3)/2 = -6.$$

Solution of Triangular Systems. Back substitution.

Solution of Upper Triangular Systems (Row-Oriented Version).

Input: Upper triangular matrix $U \in \mathbb{R}^{n \times n}$, right hand side vector $b \in \mathbb{R}^n$.

Output: Solution $x \in \mathbb{R}^n$ of $Ux = b$.

Mathematically,

$$x_i = \left(b_i - \sum_{j=i+1}^n u_{ij}x_j \right) / u_{ii}, \quad \text{if } u_{ii} \neq 0.$$

MATLAB code that overwrites b with the solution to $Ux = b$.

```
if all(diag(u)) == 0
    disp('the matrix is singular')
else
    b(n) = b(n)/U(n,n);
    for i = n-1:-1:1
        b(i) = (b(i) - U(i,i+1:n)*b(i+1:n))/U(i,i);
    end
end
```

Solution of Triangular Systems. Back substitution.

Solution of Upper Triangular Systems (Column-Oriented Version).

Input: Upper triangular matrix $U \in \mathbb{R}^{n \times n}$, right hand side vector $b \in \mathbb{R}^n$.

Output: Solution $x \in \mathbb{R}^n$ of $Ux = b$.

MATLAB code that overwrites b with the solution to $Ux = b$.

```
if all(diag(u)) == 0
    disp('the matrix is singular')
else
    for j = n:-1:2
        b(j) = b(j)/U(j,j) ;
        b(1:j-1) = b(1:j-1) - b(j)*U(1:j-1,j);
    end

    b(1) = b(1)/U(1,1);
end
```

Gaussian Elimination

Gaussian elimination for the solution of a linear system transforms the system $Ax = b$ into an equivalent system with upper triangular matrix.

This is done by applying three types of transformations to the augmented matrix $(A|b)$.

- ▶ Type 1: Replace an equation with the sum of the same equation and a multiple of another equation;
- ▶ Type 2: Interchange two equations; and
- ▶ Type 3: Multiply an equation by a nonzero number.

Once the augmented matrix $(A|b)$ is transformed into $(U|y)$, where U is an upper triangular matrix, we can use the techniques discussed previously to solve this transformed system $Ux = y$.

Gaussian Elimination (cont.)

Consider a linear system $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

The augmented system is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & 1 \end{array} \right).$$

Step 1.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 2 \\ 0 & 4 & 1 & 1 \end{array} \right) \begin{array}{l} (1) \\ (2) \leftarrow (2) - 2 * (1) \\ (3) \leftarrow (3) + 1 * (2). \end{array}$$

Gaussian Elimination (cont.)

Consider a linear system $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

The augmented system is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & 1 \end{array} \right).$$

Step 2.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 11/3 & 11/3 \end{array} \right) \begin{array}{l} (1) \\ (2) \\ (3) \leftarrow (3) + 4/3 * (2). \end{array}$$

Solving the triangular system we obtain $x_3 = 1$, $x_2 = 0$, $x_1 = 1$.

Gaussian Elimination

We need to modify Gaussian elimination for two reasons:

- ▶ improve numerical stability (change how we perform pivoting)
- ▶ make it more versatile (leads to LU-decomposition)

Gaussian Elimination (cont.)

Partial pivoting: In step i , find row j with $j > i$ such that $|a_{ji}| \geq |a_{ki}|$ for all $k > i$ and exchange rows i and j . Such numbers a_{ji} we call pivots. Again consider the augmented system

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & 1 \end{array} \right).$$

Step 1.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 2 & 2 & 1 \end{array} \right) \quad \begin{array}{l} (1) \leftarrow (2) \\ (2) \leftarrow (1) \\ (3). \end{array}$$

Gaussian Elimination (cont.)

Partial pivoting: In step i , find row j with $j > i$ such that $|a_{ji}| \geq |a_{ki}|$ for all $k > i$ and exchange rows i and j . Such numbers a_{ji} we call pivots. Again consider the augmented system

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & 1 \end{array} \right).$$

Step 2.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ 0 & 3/2 & -1 & -1 \\ 0 & 5/2 & 2 & 2 \end{array} \right) \quad \begin{array}{l} (1) \\ (2) \leftarrow (2) - 1/2 * (1) \\ (3) \leftarrow (3) - (-1/2) * (2). \end{array}$$

Gaussian Elimination (cont.)

Partial pivoting: In step i , find row j with $j > i$ such that $|a_{ji}| \geq |a_{ki}|$ for all $k > i$ and exchange rows i and j . Such numbers a_{ji} we call pivots. Again consider the augmented system

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & 1 \end{array} \right).$$

Step 3.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ 0 & 5/2 & 2 & 2 \\ 0 & 3/2 & -1 & -1 \end{array} \right) \quad \begin{array}{l} (1) \\ (2) \leftarrow (3) \\ (3) \leftarrow (2). \end{array}$$

Gaussian Elimination (cont.)

Partial pivoting: In step i , find row j with $j > i$ such that $|a_{ji}| \geq |a_{ki}|$ for all $k > i$ and exchange rows i and j . Such numbers a_{ji} we call pivots. Again consider the augmented system

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & 1 \end{array} \right).$$

Step 4.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ 0 & 5/2 & 2 & 2 \\ 0 & 0 & -\frac{11}{8} & -\frac{11}{8} \end{array} \right) \begin{array}{l} (1) \\ (2) \\ (3) \leftarrow (3) - \frac{3/2}{5/2} * (2). \end{array}$$

Solving the triangular system we obtain $x_3 = 1$, $x_2 = 0$, $x_1 = 1$.

Gaussian Elimination

Why do we need partial pivoting?

It doesn't matter if we use exact arithmetic. Without partial pivoting with floating point arithmetic the method can be unstable

How expensive is Gaussian elimination?

Using

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$$

we can calculate that for large n the number of flops in the Gaussian elimination with partial pivoting approximately equal to $2n^3/3$.