

Final Project 2.

1. (30 points)

Solve the system of linear equations

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad t > 0, \quad \mathbf{x}(0) = [1, 1, 2]^T, \quad (1)$$

where

$$A = \begin{pmatrix} -2 & 6 & 6 \\ 1 & -3 & 1 \\ -1 & -5 & -9 \end{pmatrix}$$

- First find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of the matrix A .
- Form the matrix

$$V = (\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3),$$

and compute V^{-1} .

- Show that $V^{-1}AV = D$, where D is a diagonal matrix with the eigenvalues on the diagonal.
- By making the substitution

$$\mathbf{y}(t) = V^{-1}\mathbf{x}(t).$$

solve the original system.

2. (10 points) Write a Forward Euler method for the equation (1). How large the maximum time step Δt can be for the method to be stable? Explain.
3. (20 points) Consider the following two-point boundary value problem

$$\begin{aligned} Lu &:= -u''(x) + bu'(x) = f(x) \quad x \in (0, 1), \\ u(0) &= u_0, u(1) = u_1, \end{aligned} \quad (2)$$

where the coefficient $b = b(x)$ is a smooth function.Introduce $N + 1$ nodal points

$$0 = x_0 < x_1 < \cdots < x_N = 1$$

by setting

$$x_j = jh, \quad j = 0, \dots, N,$$

where $h = 1/N$. Denote the approximation of $u(x_j)$ by U_j , $b(x_j)$ by b_j , and $f(x_j)$ by f_j . Define the following finite difference approximations of derivatives by

$$D_h U_j = \frac{U_{j+1} - U_{j-1}}{2h}, \quad D_h^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}.$$

Define a finite difference approximation of (2) by

$$\begin{aligned} L_h U_j &:= -D_h^2 U_j + b_j D_h U_j = f_j, \quad \text{for } j = 1, \dots, N-1, \\ U_0 &= u_0, \quad U_N = u_1. \end{aligned}$$

The equation at the interior point x_j may be written as

$$2U_j - (1 - \frac{hb_j}{2})U_{j+1} - (1 + \frac{hb_j}{2})U_{j-1} = h^2 f_j.$$

Assume that h is so small that $1 \pm \frac{hb_j}{2} \geq 0$ and that U satisfies $L_h U_j \geq 0$. Prove that

$$\min_{1 \leq j \leq N-1} U_j = \min \{U_0, U_N\}.$$

4. (20 points) Consider the following homogeneous heat equation

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad 0 < x < 1, \quad t > 0 \\ u(0, t) &= u(1, t) = 0, \quad t > 0, \\ u(x, 0) &= g(x).\end{aligned}$$

Show that for any $T > 0$

$$\int_0^1 u^2(x, T) dx \leq \int_0^1 g^2(x) dx.$$

Hint: Multiply the heat equation by u , integrate over the interval $(0, 1)$, integrate by parts, and integrate from 0 to T in time.

5. (20 points) Substitute the true solution $u(x, t)$ into the Lax-Friedrichs method

$$\frac{U_j^{n+1} - \frac{1}{2}(U_{j+1}^n + U_{j-1}^n)}{\Delta t} = c \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$$

and use $u_t = cu_x$ and $u_{tt} = c^2 u_{xx}$ to find the coefficient of the numerical dissipation u_{xx} .