MATH 2410 – Differential Equations April 26, 2018

Practice Final Exam 2

No calculators. Show your work. Clearly mark each answer.

1. Consider the autonomous differential equation
$$y' = (y+1)(y-3)^3(y-5)^2.$$
   (a) Compute the equilibrium solutions.
   (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.
   (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 2$.

2. A 400-gallon tank initially contains 200 gallons of sugar water at concentration of 0.1 pounds of sugar per gallon. Suppose water containing 0.5 sugar per gallon flows into the top of the tank at a rate of 2 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at rate 1 gallon per minute. How much sugar is in the tank when the tank is full?

3. Solve the initial value problem
$$\begin{aligned} y' + \frac{3y}{t+1} &= (t+1)^2 \\ y(0) &= 3. \end{aligned}$$

4. The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.
$$\begin{aligned} \frac{dx}{dt} &= x(2-x-y) \\ \frac{dy}{dt} &= y(6-2x-2y). \end{aligned}$$
Draw the curves $x(t)$ and $y(t)$ if $x(0) = 3$ and $y(0) = 3$ in the phase plane.

5. Compute the Euler's approximate solution at time $t = 1$ of the following system
$$\begin{aligned} \frac{dx}{dt} &= x(2-2x-y) \\ \frac{dy}{dt} &= y(t-x-2y). \end{aligned}$$
With initial position $x(0) = 2$ and $y(0) = 1$ and time step $\Delta t = 0.5$

6. Consider the linear system $\vec{Y}' = A\vec{Y}$, where
$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -4 & -4 \\ -6 & -2 \end{pmatrix}$$
   (a) Compute the eigenvalues of A.
   (b) Classify the equilibrium at the origin (sink, spiral source, etc). Explain your answer.

(c) What is the general solution to the system? Sketch the phase plane.

7. Compute the general solution to the linear system $\vec{Y}' = A\vec{Y}$, where

$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}$$

Sketch the phase plane.

8. Consider the spring-mass system whose motion is governed by

$$y'' + 4y' + 5y = 2 - t.$$

(a) Compute the solution to the above equation if $y(0) = 0$, $y'(0) = 0$.

(b) Describe (in words) the long term behavior of the mass.

9. Find the general solution for the damped spring-mass problem

$$y'' + 4y = \cos(2t).$$

Solve with initial conditions $y(0) = 0$, $y'(0) = 1$.

10. Consider the equation

$$y' + 6y = e^{-2t}$$

with initial conditions $y(0) = 1$. Using the Laplace transform, find $y(t)$.

11. Consider the equation

$$y' + 8y = 2 + H_3(t)$$

with initial conditions $y(0) = 0$, where $H_3(t)$ is the Heavyside function,

$$H_3(t) = \begin{cases} 0, & 0 \le t < 3 \\ 1, & t \ge 3. \end{cases}$$

Using the Laplace transform, find $y(t)$.