Practice Final Exam 2. Solutions.

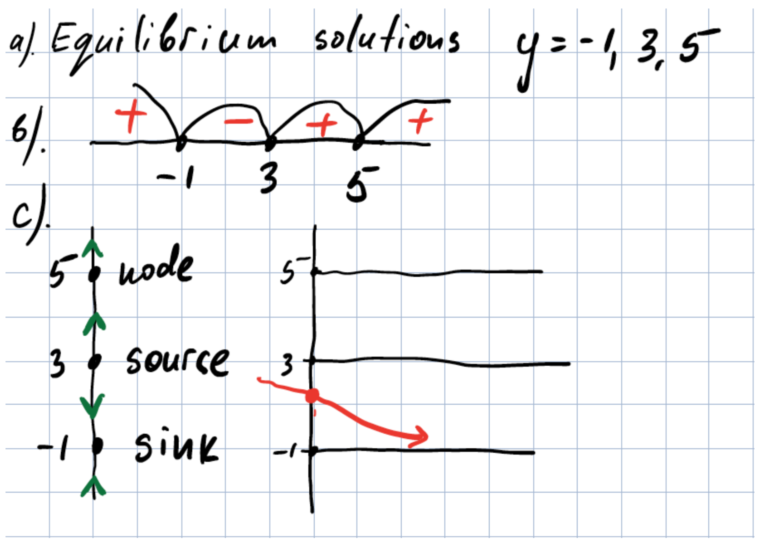
No calculators. Show your work. Clearly mark each answer.

1. Consider the autonomous differential equation

$$y' = (y+1)(y-3)^3(y-5)^2.$$

   (a) Compute the equilibrium solutions.
   (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.
   (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 2$.

Solution:



2. A 400-gallon tank initially contains 200 gallons of sugar water at concentration of 0.1 pounds of sugar per gallon. Suppose water containing 0.5 sugar per gallon flows into the top of the tank at a rate of 2 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at rate 1 gallon per minute. How much sugar is in the tank when the tank is full?

Solution:

Let $c(t)$ be amount of sugar in the tank. Then the concentration is $C(t) = \frac{c(t)}{V(t)}$, where is $V(t)$ is the volume of the fluid in the tank, which changes in time. Thus, $C(0) = 0.1$ $lb./gal.$ and $c(0) = 0.1$ $lb./gal. * 200$ $gal. = 20$ $lb.$ and we need to find $C(200) = \frac{c(200)}{400}$, since $V(200) = 400$. The sugar enters the tank at the rate of

$$\frac{1}{2}\ lb./gal. \times 2\ gal./min. = 1\ lb./min.$$

and leaves at the rate of

$$\frac{c(t)}{200+t}\ lb./gal. \times 1\ gal./min. = \frac{c(t)}{200+t}\ lb./min.$$

Thus, the rate of change rate for the amount of the mixture is

$$c'(t) = 1 - \frac{c(t)}{200+t}.$$

Hence the equation for the amount of the mixture is

$$c'(t) + \frac{c(t)}{200+t} = 1, \quad c(0) = 20.$$

The integration factor is $1+t$ hence

$$\frac{d}{dt}\left(c(t)(200+t)\right) = 200+t$$

integration both sides we obtain

$$c(t)(200+t) = \frac{(200+t)^2}{2} + A, \quad \text{for any constant } A.$$

From the condition $c(0) = 20$ we find

$$20 * 200 = \frac{(200)^2}{2} + A \quad \Longrightarrow \quad A = 200(20-100) = -80 * 200$$

Hence

$$c(t) = \frac{(200+t)}{2} - \frac{80*200}{200+t},$$

and as a result

$$c(200) = \frac{(200+200)}{2} - \frac{80*200}{200+200} = 200 - 40 = 160 \; lb.$$

and $\boxed{C(200) = \frac{c(200)}{400} = \frac{160}{400} = 0.4 \; lb./gal.}$

3. Solve the initial value problem

$$y' + \frac{3y}{t+1} = (t+1)^2$$
$$y(0) = 3.$$

Solution:

The integrating factor is

$$\tau(t) = e^{\int \frac{3}{1+t} dt} = e^{3\ln(1+t)} = e^{\ln(1+t)^3} = (1+t)^3.$$

Multiplying by the integrating factor $\tau(t)$, we obtain

$$\frac{d}{dt}\left((1+t)^3 y(t)\right) = (1+t)^5.$$

Integrating both sides, we obtain

$$(1+t)^3 y(t) = \frac{(1+t)^6}{6} + C.$$

From the condition $y(0) = 3$, we find

$$y(0) = 3 = \frac{1}{6} + C \quad \Rightarrow \quad C = \frac{17}{6}.$$

As a result

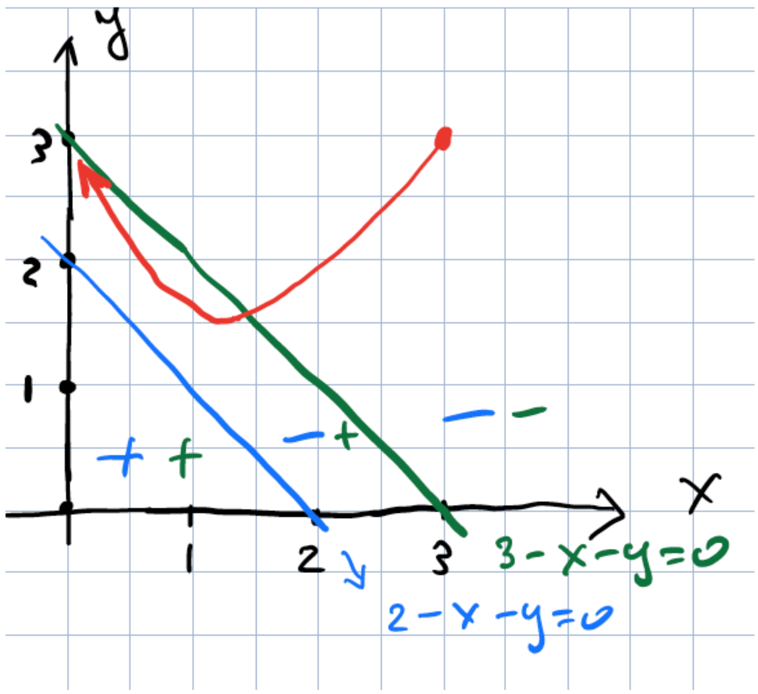
$$\boxed{y(t) = \frac{(1+t)^3}{6} + \frac{17}{6}\frac{1}{(1+t)^3}.}$$

4. The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(2 - x - y)$$
$$\frac{dy}{dt} = y(6 - 2x - 2y).$$

Draw the curves $x(t)$ and $y(t)$ if $x(0) = 3$ and $y(0) = 3$ in the phase plane.

Solution:



5. Compute the Euler's approximate solution at time $t = 1$ of the following system

$$\frac{dx}{dt} = x(2 - 2x - y)$$
$$\frac{dy}{dt} = y(t - x - 2y).$$

With initial position $x(0) = 2$ and $y(0) = 1$ and time step $\Delta t = 0.5$

Solution: We can rewrite the above system in vector form

$$\vec{z}'(t) = \vec{F}(t, \vec{z}(t)),$$

where $\vec{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\vec{F}(t, \vec{z}(t)) = \begin{pmatrix} x(2 - 2x - y) \\ y(t - x - 2y) \end{pmatrix}$ and $\vec{z}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The Euler's method for this problem is

$$\vec{z}^{n+1} = \vec{z}^n + \Delta t \cdot \vec{F}(t_n, \vec{z}^n), \quad n = 0, 1, \dots.$$

Since $\Delta t = 0.5$, in order to approximate $\vec{z}(1)$, we only need two steps of the method. Since $\vec{z}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t_0 = 0$, we find

$$\begin{aligned} \vec{z}^1 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \Delta t \cdot \begin{pmatrix} x(0)(2 - 2x(0) - y(0)) \\ y(0)(t_0 - x(0) - 2y(0)) \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 2(2-4-1) \\ 1(0-2-2) \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \end{aligned}$$

Thus $x^1 = -1$ and $y^1 = -1$, and $t_1 = \frac{1}{2}$. Hence

$$\begin{aligned} \vec{z}^2 = \begin{pmatrix} x^1 \\ y^1 \end{pmatrix} + \Delta t \cdot \begin{pmatrix} x^1(2 - 2x^1 - y^1) \\ y^1(t_1 - x^1 - 2y^1) \end{pmatrix} &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} -1(2+2+1) \\ -1(1/2+1+1) \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -5 \\ -5/2 \end{pmatrix} = \begin{pmatrix} -7/2 \\ -9/4 \end{pmatrix}. \end{aligned}$$

Thus $\boxed{x(1) \approx x^2 = -3.5}$ and $\boxed{y(1) \approx y^2 = -2.25}$.

6. Consider the linear system $\vec{Y}' = A\vec{Y}$, where

$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -4 & -4 \\ -6 & -2 \end{pmatrix}$$

(a) Compute the eigenvalues of A.

(b) Classify the equilibrium at the origin (sink, spiral source, etc). Explain your answer.

(c) What is the general solution to the system? Sketch the phase plane.

Solution:

The characteristic polynomial is

$$det \begin{pmatrix} -4-\lambda & -4 \\ -6 & -2-\lambda \end{pmatrix} = (-4-\lambda)(-2-\lambda) - 24 = \lambda^2 + 6\lambda - 16 = (\lambda+8)(\lambda-2).$$

Hence the matrix A has two real eigenvalues $\boxed{\lambda_1 = -8 \text{ and } \lambda_2 = 2}$. Since we have two real eigenvectors, one positive and one negative the equilibrium is a node (saddle).

In order to find straight line solution method we need to find the corresponding eigenvectors.

For $\lambda_1 = -8$ we have

$$\begin{pmatrix} -4+8 & -4 \\ -6 & -2+8 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -6 & 6 \end{pmatrix}.$$

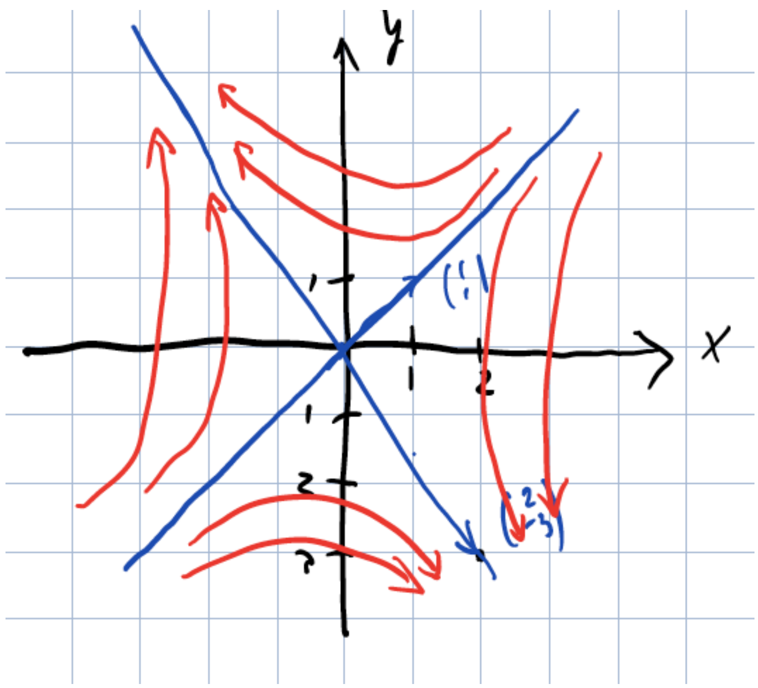
Thus the corresponding eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$ we have

$$\begin{pmatrix} -4-2 & -4 \\ -6 & -2-2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ -6 & -4 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Hence the straight line solution is

$$\boxed{\vec{Y}(t) = c_1 e^{-8t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}}, \quad c_1, c_2 \in \mathbb{R}.$$



7. Compute the general solution to the linear system $\vec{Y}' = A\vec{Y}$, where

$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}$$

Sketch the phase plane.

Solution:

The characteristic polynomial is

$$det\begin{pmatrix} -1-\lambda & 4 \\ -4 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 + 4^2.$$

Hence the matrix A has two complex eigenvalues $\lambda_1 = -1+4i$ and $\lambda_2 = -1-4i$. For $\lambda = -1+4i$ we have

$$\begin{pmatrix} -1-(-1+4i) & 4 \\ -4 & -1-(-1+4i) \end{pmatrix} = \begin{pmatrix} -4i & 4 \\ -4 & -4i \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Thus

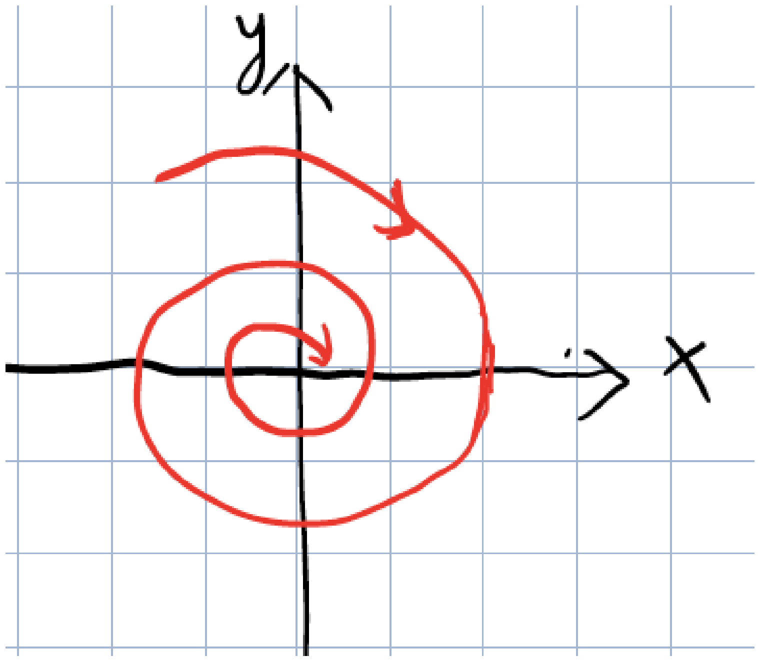
$$e^{(-1+4i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t}(\cos(4t) + i\sin(4t))\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t}\left(\begin{pmatrix} \cos(4t) \\ -\sin(4t) \end{pmatrix} + i\begin{pmatrix} \sin(4t) \\ \cos(4t) \end{pmatrix}\right)$$

As a result the general solution is

$$\boxed{c_1e^{-t}\begin{pmatrix} \cos(4t) \\ -\sin(4t) \end{pmatrix} + c_2e^{-t}\begin{pmatrix} \sin(4t) \\ \cos(4t) \end{pmatrix}}$$

8. Consider the spring-mass system whose motion is governed by

$$y'' + 4y' + 5y = 2 - t.$$



(a) Compute the solution to the above equation if $y(0) = 0$, $y'(0) = 0$.

(b) Describe (in words) the long term behavior of the mass.

Solution.

a). First we consider the homogeneous problem

$$y'' + 4y' + 5y = 0.$$

The corresponding characteristic equation is

$$r^2 + 4r + 5 = (r+2)^2 + 1 = 0 \quad \Longrightarrow \quad r = -2 \pm i.$$

Since the roots are complex, the general solution to the homogeneous problem is

$$y_H = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

The particular solution is of the form $y_p = A + Bt$. Inserting it into the equation we obtain

$$4B + 5A + 5Bt = 2 - t \quad \Longrightarrow \quad 5B = -1 \quad \Longrightarrow \quad B = -\frac{1}{5}; \quad \Longrightarrow \quad 4B + 5A = 2 \quad \Longrightarrow \quad A = \frac{14}{25}.$$

Thus the general solution is

$$y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t + \frac{14}{25} - \frac{t}{5}$$

and as a result

$$0 = y(0) = c_1 + 0 + \frac{14}{25} \quad \Longrightarrow \quad c_1 = -\frac{14}{25}$$

Differentiating, we find

$$y'(t) = -3c_1 e^{-3t} \cos(5t) - 5c_1 e^{-3t} \sin(5t) - 3c_2 e^{-3t} \sin(5t) + 5c_2 e^{-3t} \cos(5t) - \frac{2}{29} e^{-t}$$

and as a result

$$0 = y'(0) = -3c_1 + 5c_2 - \frac{2}{29} \quad \Longrightarrow \quad 5c_2 = 3c_1 + \frac{2}{29} \quad \Longrightarrow \quad c_2 = -\frac{6}{145} + \frac{2}{145} = -\frac{4}{145}.$$

Thus the solution to initial value problem is

$$\boxed{y(t) = -\frac{2}{29}e^{-3t}\cos(5t) - \frac{4}{145}e^{-3t}\sin(5t) + \frac{2}{29}e^{-t}.}$$

b). Since e^{-t} dominates e^{-3t} for t sufficiently large, the solution starting at $(0, 0)$ after ossilationg a little bit will be almost indistinguishable from e^{-t} and of course approaches zero at $t \to \infty$.

9. Find the general solution for the damped spring-mass problem

$$y'' + 4y = \cos(2t).$$

Solve with initial conditions $y(0) = 0$, $y'(0) = 1$.

Solutions: Since the equation is linear, the general solution is of the form,

$$y(t) = y_H(t) + y_P(t),$$

where y_H is the general solution to homogeneous problem $y''+4y = 0$ and y_P is any particular solution. Looking $y_H(t)$ of the form e^{st}, we easily find that s must satisfy

$$s^2 + 4 = 0 \quad \Longrightarrow \quad s = \pm 2i.$$

Hence the general solution to the homogeneous equation is

$$y_H(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

To look for a particular solution, we use that $\sin(2t) = Im(e^{2it})$. Since $\sin(2t)$ is part of the homogeneous solution, the particular solution we are looking for is of the form

$$y_P(t) = Cte^{2it}.$$

Since

$$y_P'(t) = Ce^{2it} + 2Cite^{2it}, \quad y_P''(t) = 4Cie^{2it} - 4Cte^{2it},$$

we have

$$y_P''(t) + 4y_P(t) = 4Cie^{2it} - 4Cte^{2it} + 4Cte^{2it} = 4Cie^{2it} = e^{2it}.$$

Thus, we find that $C = \frac{1}{4i} = -\frac{i}{4}$ and as a result

$$y_P(t) = Re\left(-\frac{i}{4}te^{2it}\right) = Re\left(-\frac{i}{4}t[\cos(2t) + i\sin(2t)]\right) = \frac{t}{4}\sin(2t).$$

Hence the general solution is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{t}{4}\sin(2t).$$

To find the constants c_1 and c_2, we use the initial conditions $y(0) = 0$ and $y'(0) = 0$. The first initial condition gives us $0 = y(0) = c_1$. Thus the solution reduces to

$$y(t) = c_2 \sin(2t) + \frac{t}{4}\sin(2t).$$

Since

$$y'(t) = 2c_2 \cos(2t) + \frac{1}{4}\sin(2t) + \frac{t}{2}\cos(2t),$$

the second initial condition gives us

$$1 = y'(0) = 2c_2 \quad \Longrightarrow \quad \Longrightarrow \quad c_2 = \frac{1}{2}.$$

As a result finally we obtain that the solution to the initial value problem is

$$\boxed{y(t) = \frac{1}{2}\sin(2t) + \frac{t}{4}\sin(2t).}$$

10. Consider the equation

$$y' + 6y = e^{-2t}$$

with initial conditions $y(0) = 1$. Using the Laplace transform, find $y(t)$.

Solutions:

Taking Laplace transform $\mathcal{L}$ on both sides of the equation and using that

$$\mathcal{L}[e^{-2t}] = \frac{1}{s+2}$$

and

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

we obtain,

$$s\mathcal{L}[y] - 1 + 6\mathcal{L}[y] = \frac{1}{s+2}.$$

Solving for $\mathcal{L}[y]$, we obtain

$$(s+6)\mathcal{L}[y] = \frac{1}{s+2} + 1, \quad \Longrightarrow \quad \mathcal{L}[y] = \frac{1}{(s+6)(s+2)} + \frac{1}{s+6}.$$

Using that

$$\frac{1}{(s+6)(s+2)} = \frac{1}{4}\left(\frac{1}{s+2} - \frac{1}{s+6}\right),$$

inverting the Laplace transform and using that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$, we obtain

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{4}\frac{1}{(s+2)} + \frac{3}{4}\frac{1}{(s+6)}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s+6}\right] = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}.$$

Thus,

$$\boxed{y(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}}$$

11. Consider the equation

$$y' + 8y = 2 + H_3(t)$$

with initial conditions $y(0) = 0$, where $H_3(t)$ is the Heavyside function,

$$H_3(t) = \begin{cases} 0, & 0 \le t < 3 \\ 1, & t \ge 3. \end{cases}$$

Using the Laplace transform, find $y(t)$.

Solutions:

Taking Laplace transform $\mathcal{L}$ on both sides of the equation and using that

$$\mathcal{L}[2] = \frac{2}{s} \quad , \mathcal{L}[H_3] = \frac{e^{-3s}}{s}, \quad \text{and} \quad \mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

we obtain,

$$s\mathcal{L}[y] + 8\mathcal{L}[y] = \frac{1}{s^2} + \frac{e^{-3s}}{s}.$$

Solving for $\mathcal{L}[y]$, we obtain

$$(s+8)\mathcal{L}[y] = \frac{2}{s} + \frac{e^{-3s}}{s}, \quad \Longrightarrow \quad \mathcal{L}[y] = \frac{2}{s(s+8)} + \frac{e^{-3s}}{s(s+8)}.$$

Using that

$$\frac{1}{s(s+8)} = \frac{1}{8}\left(\frac{1}{s} - \frac{1}{s+8}\right),$$

inverting the Laplace transform and using that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$, and $\mathcal{L}[H_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$, we obtain

$$\begin{aligned} y(t) &= \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+8}\right] + \frac{1}{8}\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s+8}\right] \\ &= \frac{1}{8} - \frac{1}{8}e^{-8t} + \frac{1}{8}H_3(t) - \frac{1}{8}H_3(t)e^{-8(t-3)}. \end{aligned}$$

Thus,

$$\boxed{y(t) = \frac{1}{8}\left(1 - e^{-8t}\right) + \frac{1}{8}H_3(t)\left(1 - e^{-8(t-3)}\right)}$$