MATH 2410 – Differential Equations April 25, 2017

Practice Final Exam. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. Sketch the slope filed of the autonomous differential equation

$$y' = (y+1)(y-3)^2(y-5).$$

Solution: Discussed in class.

2. Sketch the slope filed of the following differential equation

$$y' = x - y.$$

Solution: Discussed in class.

3. A five gallon tank has 1 gallon of pure water. We open a spigot so 1 gal. leaves the tank and introduce a mixture of 1/2 lb. per gal at 2 gal per minute. Assuming the mixture is well mixed, what is the concentration at the time when the tank is full?

Solution:

Let $c(t)$ be amount of the mixture in the tank. Then the concentration is $C(t) = \frac{c(t)}{V(t)}$, where is $V(t)$ is the volume of the fluid in the tank, which changes in time. Thus, $C(0) = 0$ and we need to find $C(4) = \frac{c(4)}{5}$, since $V(4) = 5$. The mixture enters the tank at the rate of

$$\frac{1}{2}\ lb./gal. \times 2\ gal./min. = 1\ lb./min.$$

and leaves at the rate of

$$\frac{c(t)}{1+t}\ lb./gal. \times 1\ gal./min. = \frac{c(t)}{1+t}\ lb./min.$$

Thus, the rate of change rate for the amount of the mixture is

$$c'(t) = 1 - \frac{c(t)}{1+t}.$$

Hence the equation for the amount of the mixture is

$$c'(t) + \frac{c(t)}{1+t} = 1, \quad c(0) = 0.$$

The integration factor is $1 + t$ hence

$$\frac{d}{dt}\left(c(t)(1+t)\right) = 1 + t$$

integration both sides we obtain

$$c(t)(1+t) = t + \frac{t^2}{2} + A, \quad \text{for any constant } A.$$

From the condition $c(0) = 0$ we find that $A = 0$. Hence

$$c(t) = \frac{t^2/2 + t}{1 + t},$$

and as a result $\boxed{C(4) = \frac{c(4)}{5} = \frac{8+4}{(1+4)5} = \frac{12}{25}\ lb./gal.}$

4. Solve the initial value problem

$$y' - \frac{3y}{t+1} = (t+1)^2$$
$$y(0) = 3.$$

Solution:

The integrating factor is

$$\tau(t) = e^{-\int \frac{3}{1+t}dt} = e^{-3\ln(1+t)} = e^{\ln(1+t)^{-3}} = (1+t)^{-3}.$$

Multiplying by the integrating factor $\tau(t)$, we obtain

$$\frac{d}{dt}\left(\frac{y(t)}{(1+t)^3}\right) = \frac{1}{1+t}.$$

Integrating both sides, we obtain

$$\frac{y(t)}{(1+t)^3} = \ln(1+t) + C.$$

From the condition $y(0) = 3$, we find

$$\frac{y(0)}{(1+0)^3} = 3 = \ln(1+0) + C \quad \Rightarrow \quad C = 3.$$

As a result

$$\boxed{y(t) = (1+t)^3\left(\ln(1+t) + 3\right).}$$

5. The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(2 - 3x - y).$$

Draw the curves $x(t)$ and $y(t)$ if $x(0) = 10$ and $y(0) = 1$ in the phase plane. **Solution:** Discussed in class.

6. Compute the Euler's approximate solution at time $t = 1$ of the following system

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(1 - x - 2y).$$

With initial position $x(0) = 2$ and $y(0) = 1$ and time step $\Delta t = 0.5$.

Solution: We can rewrite the above system in vector form

$$\vec{z}'(t) = \vec{F}(\vec{z}(t)),$$

where $\vec{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\vec{F}(\vec{z}(t)) = \begin{pmatrix} x(1-x-y) \\ y(1-x-2y) \end{pmatrix}$ and $\vec{z}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
The Euler's method for this problem is

$$\vec{z}^{n+1} = \vec{z}^n + \Delta t \cdot \vec{F}(\vec{z}^n), \quad n = 0, 1, \ldots.$$

Since $\Delta t = 0.5$, in order to approximate $\vec{z}(1)$, we only need two steps of the method. Since $\vec{z}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we find

$$\begin{aligned}\vec{z}^1 &= \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \Delta t \cdot \begin{pmatrix} x(0)(1-x(0)-y(0)) \\ y(0)(1-x(0)-2y(0)) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 2(1-2-1) \\ 1(1-2-2) \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.\end{aligned}$$

Thus $x^1 = 0$ and $y^1 = -1/2$. Hence

$$\begin{aligned}\vec{z}^2 &= \begin{pmatrix} x^1 \\ y^1 \end{pmatrix} + \Delta t \cdot \begin{pmatrix} x^1(1-x^1-y^1) \\ y^1(1-x^1-2y^1) \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0(1+0+1/2) \\ -1/2(1+0+1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.\end{aligned}$$

Thus $\boxed{x(1) \approx x^2 = 0}$ and $\boxed{y(1) \approx y^2 = -1}$.

7. Find the solution to the following linear system

$$\begin{aligned}\frac{dx}{dt} &= 2 - 2x \\ \frac{dy}{dt} &= -x - 2y\end{aligned}$$

with initial position $x(0) = 1$ and $y(0) = 1$.

Solution:

The system is decoupled. First we solve the equation for $x(t)$. Thus

$$x(t) = x_H(t) + x_p(t),$$

where $x_H(t)$ is the general solution to the homogeneous problem $x'(t) = -2x(t)$ and $x_p(t)$ is any particular solution. Thus we find that

$$x_H(t) = Ce^{-2t}, \quad x_p(t) = 1.$$

As a result

$$x(t) = Ce^{-2t} + 1,$$

and from the condition $x(0) = 1$ we find that $C = 0$ and hence $\boxed{x(t) = 1}$. Using that we now know that $x(t) = 1$, the equation for $y(t)$ becomes

$$y'(t) = -1 - 2y, \quad y(0) = 1.$$

Again the solution is of the form

$$y(t) = y_H(t) + y_p(t),$$

where $y_H(t)$ is the general solution to the homogeneous problem $y'(t) = -yx(t)$ and $y_p(t)$ is any particular solution. Thus we find that

$$y_H(t) = Ce^{-2t}, \quad y_p(t) = -1/2.$$

As a result

$$y(t) = Ce^{-2t} - \frac{1}{2},$$

and from the condition $y(0) = 1$ we find that $C = \frac{3}{2}$ and hence $\boxed{y(t) = \frac{3}{2}e^{-2t} - \frac{1}{2}}$.

8. Consider the following second order equation

$$y'' + 6y' + 34y = 2e^{-t}.$$

(a) Compute the solution to the above equation if $y(0) = 0$, $y'(0) = 0$.

(b) Describe (in words) the long term behavior of the mass.

Solution.

a). First we consider the homogeneous problem

$$y'' + 6y' + 34y = 0.$$

The corresponding characteristic equation is

$$r^2 + 6r + 34 = (r+3)^2 + 5^2 = 0 \quad \Longrightarrow \quad r = -3 \pm 5i.$$

Since the roots are complex, the general solution to the homogeneous problem is

$$y_H = c_1 e^{-3t}\cos(5t) + c_2 e^{-3t}\sin(5t).$$

The particular solution is of the form $y_p = Ae^{-t}$. Inserting it into the equation we obtain

$$A - 6A + 34A = 2 \quad \Longrightarrow \quad 29A = 2 \quad \Longrightarrow \quad A = \frac{2}{29}.$$

Thus the general solution is

$$y(t) = c_1 e^{-3t}\cos(5t) + c_2 e^{-3t}\sin(5t) + \frac{2}{29}e^{-t}$$

and as a result

$$0 = y(0) = c_1 + 0 + \frac{2}{29} \quad \Longrightarrow \quad c_1 = -\frac{2}{29}$$

Differentiating, we find

$$y'(t) = -3c_1 e^{-3t}\cos(5t) - 5c_1 e^{-3t}\sin(5t) - 3c_2 e^{-3t}\sin(5t) + 5c_2 e^{-3t}\cos(5t) - \frac{2}{29}e^{-t}$$

and as a result

$$0 = y'(0) = -3c_1 + 5c_2 - \frac{2}{29} \quad \Longrightarrow \quad 5c_2 = 3c_1 + \frac{2}{29} \quad \Longrightarrow \quad c_2 = -\frac{6}{145} + \frac{2}{145} = -\frac{4}{145}.$$

Thus the solution to initial value problem is

$$\boxed{y(t) = -\frac{2}{29}e^{-3t}\cos(5t) - \frac{4}{145}e^{-3t}\sin(5t) + \frac{2}{29}e^{-t}.}$$

b). Since e^{-t} dominates e^{-3t} for t sufficiently large, the solution starting at $(0,0)$ after ossilationg a little bit will be almost indistinguishable from e^{-t} and of course approaches zero at $t \to \infty$.

9. Find the general solution for the damped spring-mass problem

$$y'' + 4y = \sin(2t).$$

Solve with initial conditions $y(0) = 0$, $y'(0) = 1$.

Solutions: Since the equation is linear, the general solution is of the form,

$$y(t) = y_H(t) + y_P(t),$$

where y_H is the general solution to homogeneous problem $y'' + 4y = 0$ and y_P is any particular solution. Looking $y_H(t)$ of the form e^{st}, we easily find that s must satisfy

$$s^2 + 4 = 0 \quad \Longrightarrow \quad s = \pm 2i.$$

Hence the general solution to the homogeneous equation is

$$y_H(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

To look for a particular solution, we use that $\sin(2t) = Im(e^{2it})$. Since $\sin(2t)$ is part of the homogeneous solution, the particular solution we are looking for is of the form

$$y_P(t) = Cte^{2it}.$$

Since

$$y_P'(t) = Ce^{2it} + 2Cite^{2it}, \quad y_P''(t) = 4Cie^{2it} - 4Cte^{2it},$$

we have

$$y_P''(t) + 4y_P(t) = 4Cie^{2it} - 4Cte^{2it} + 4Cte^{2it} = 4Cie^{2it} = e^{2it}.$$

Thus, we find that $C = \frac{1}{4i} = -\frac{i}{4}$ and as a result

$$y_P(t) = Im\left(-\frac{i}{4}te^{2it}\right) = Im\left(-\frac{i}{4}t(\cos(2t) + i\sin(2t)\right) = -\frac{t}{4}\cos(2t).$$

Quick check:

$$y_P'(t) = -\frac{1}{4}\cos(2t) + \frac{t}{2}\sin(2t), \quad y_P''(t) = \sin(2t) + t\cos(2t),$$

and

$$y_P''(t) + 4y_P(t) = \sin(2t) + t\cos(2t) - 4\frac{t}{4}\cos(2t) = \sin(2t).$$

Hence the general solution is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{t}{4}\cos(2t).$$

To find the constants c_1 and c_2, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. The first initial condition gives us $0 = y(0) = c_1$. Thus the solution reduces to

$$y(t) = c_2 \sin(2t) - \frac{t}{4}\cos(2t).$$

Since

$$y'(t) = 2c_2 \cos(2t) - \frac{1}{4}\cos(2t) + \frac{t}{2}\sin(2t),$$

the second initial condition gives us

$$1 = y'(0) = 2c_2 - \frac{1}{4} \quad \Longrightarrow \quad 2c_2 = \frac{1}{4} + 1 = \frac{5}{4} \quad \Longrightarrow \quad c_2 = \frac{5}{8}.$$

As a result finally we obtain that the solution to the initial value problem is

$$\boxed{y(t) = \frac{5}{8}\sin(2t) - \frac{t}{4}\cos(2t).}$$

10. Using the Laplace transform solve the following initial value problem

$$y' + 6y = e^{-2t} + 2, \quad y(0) = 2.$$

Solution:

Taking Laplace transform $\mathcal{L}$ on both sides of the equation and using that

$$\mathcal{L}[e^{-2t}] = \frac{1}{s+2}$$

and

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

we obtain,

$$s\mathcal{L}[y] - 1 + 6\mathcal{L}[y] = \frac{1}{s+2}.$$

Solving for $\mathcal{L}[y]$, we obtain

$$(s+6)\mathcal{L}[y] = \frac{1}{s+2} + 1, \quad \Longrightarrow \quad \mathcal{L}[y] = \frac{1}{(s+6)(s+2)} + \frac{1}{s+6}.$$

Using that

$$\frac{1}{(s+6)(s+2)} = \frac{1}{4}\left(\frac{1}{s+2} - \frac{1}{s+6}\right),$$

inverting the Laplace transform and using that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$, we obtain

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{4}\frac{1}{(s+2)} + \frac{3}{4}\frac{1}{(s+6)}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s+6}\right] = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}.$$

Thus,

$$\boxed{y(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}}$$

11. Using the Laplace transform solve the following initial value problem

$$y' + 9y = 1 + H_2(t), \quad y(0) = 1,$$

where $H_2(t)$ is the Heavyside function,

$$H_2(t) = \begin{cases} 0, & 0 \le t < 2 \\ 1, & t \ge 2. \end{cases}$$

Taking Laplace transform $\mathcal{L}$ on both sides of the equation and using that

$$\mathcal{L}[1] = \frac{1}{s} \quad, \mathcal{L}[H_2] = \frac{e^{-2s}}{s}, \quad \text{and} \quad \mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

we obtain,

$$s\mathcal{L}[y] + 9\mathcal{L}[y] - 1 = \frac{1}{s} + \frac{e^{-2s}}{s}.$$

Solving for $\mathcal{L}[y]$, we obtain

$$(s+9)\mathcal{L}[y] = 1 + \frac{1}{s} + \frac{e^{-2s}}{s}, \quad \Longrightarrow \quad \mathcal{L}[y] = \frac{1}{s+9} + \frac{1}{s(s+9)} + \frac{e^{-2s}}{s(s+9)}.$$

Using that

$$\frac{1}{s(s+9)} = \frac{1}{9}\left(\frac{1}{s} - \frac{1}{s+9}\right),$$

inverting the Laplace transform and using that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$, and $\mathcal{L}[H_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$, we obtain

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+9}\right] + \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+9}\right] + \frac{1}{9}\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s+9}\right] = e^{-9t} + \frac{1}{9} - \frac{1}{9}e^{-9t} + \frac{1}{9}H_2(t) - \frac{1}{9}H_2(t)e^{-9(t-2)}.$$

Thus,

$$\boxed{y(t) = e^{-9t} + \frac{1}{9}\left(1 - e^{-9t}\right) + \frac{1}{9}H_2(t)\left(1 - e^{-9(t-2)}\right)}$$