MATH 2410 – Differential Equations April 11, 2017

Practice Exam 3. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. Find the general solution of the following second order equation

$$y'' - 6y' + 9y = 0.$$

Solution. We are looking for a solution in the form $y = e^{rx}$. Insering it into the equation we obtain the characteristic equation

$$r^2 - 6r + 9 = (r-3)^2 = 0.$$

Since $r = 3$ is a double root, the general solution is

$$\boxed{y = c_1 e^{3x} + c_2 x e^{3x}.}$$

2. Find the general solution of the following second order equation

$$y'' - 6y' + 34y = 0.$$

Solution. Again we are looking for a solution in the form $y = e^{rx}$. Insering it into the equation we obtain the characteristic equation

$$r^2 - 6r + 34 = (r-3)^2 + 5^2 = 0 \quad \Longrightarrow \quad r = 3 + 5i.$$

Since the roots are complex, the general solution is

$$\boxed{y = c_1 e^{3x} \cos(5x) + c_2 e^{3x} \sin(5x).}$$

3. Find a particular solution of the following second order equation

$$y'' - 6y' + 7y = 2x + e^x.$$

Solution. First we study the homogeneous problem

$$y'' - 6y' + 7y = 0.$$

The characteristic equation is

$$r^2 - 6r + 7 = (r-3)^2 - 2 = 0 \quad \Longrightarrow \quad (r-3)^2 = 2 \quad \Longrightarrow \quad r = 3 \pm \sqrt{2}.$$

Since the roots are not equal to 1 the particular solution should be in the form

$$y_p = A + Bx + Ce^x.$$

Differentiating, we obtain

$$y_p' = B + Ce^x, \quad y_p'' = Ce^x.$$

Inserting it into the equation we obtain

$$Ce^x - 6B - 6Ce^x + 7A + 7Bx + 7Ce^x = 2x + e^x.$$

Since the coefficients in from of 1, x and e^x must match, we obtain

$$C - 6C + 7C = 1 \quad \Longrightarrow \quad 2C = 1 \quad \Longrightarrow \quad C = \frac{1}{2},$$

$$7B = 2 \quad \Longrightarrow \quad B = \frac{2}{7},$$

and

$$-6B + 7A = 0 \quad \Longrightarrow \quad 7A = 6B = \frac{12}{7} \quad \Longrightarrow \quad A = \frac{12}{49}.$$

Thus the particular solution is

$$\boxed{y_p = \frac{12}{49} + \frac{2}{7}x + \frac{1}{2}e^x.}$$

4. Find a particular solution of the following second order equation

$$y'' - 5y' + 6y = xe^x.$$

Solution. First we study the homogeneous problem

$$y'' - 5y' + 6y = 0.$$

The characteristic equation is

$$r^2 - 5r + 6 = \left(r - \frac{5}{2}\right)^2 - \frac{1}{4} = 0 \quad \Longrightarrow \quad \left(r - \frac{5}{2}\right)^2 = \frac{1}{4} \quad \Longrightarrow \quad r = \frac{5}{2} \pm \frac{1}{2}.$$

Since the roots are not equal to 1 the particular solution should be in the form

$$y_p = (A + Bx)e^x.$$

Differentiating, we obtain

$$y_p' = Be^x + (A + Bx)e^x, \quad y_p'' = 2Be^x + (A + Bx)e^x.$$

Inserting it into the equation we obtain

$$2Be^x + (A + Bx)e^x - 5Be^x - 5(A + Bx)e^x + 6(A + Bx)e^x = xe^x.$$

Canceling by e^x we obtain

$$2B + (A + Bx) - 5B - 5(A + Bx) + 6(A + Bx) = x.$$

Since the coefficients in from of 1 and x must match, we obtain

$$B - 5B + 6B = 1 \quad \Longrightarrow \quad B = \frac{1}{2}$$

and

$$2A - 3B = 0 \quad \Longrightarrow \quad 2A = 3B = \frac{3}{2} \quad \Longrightarrow \quad A = \frac{3}{4}$$

Thus the particular solution is

$$\boxed{y_p = \left(\frac{3}{4} + \frac{x}{2}\right)e^x.}$$

5. Consider the spring-mass system whose motion is governed by

$$y'' + 6y' + 34y = 2e^{-t}.$$

(a) Compute the solution to the above equation if $y(0) = 0$, $y'(0) = 0$.
Solution. First we consider the homogeneous problem

$$y'' + 6y' + 34y = 0.$$

The corresponding characteristic equation is

$$r^2 + 6r + 34 = (r+3)^2 + 5^2 = 0 \quad \Longrightarrow \quad r = -3 \pm 5i.$$

Since the roots are complex, the general solution to the homogeneous problem is

$$y_H = c_1 e^{-3t} \cos(5t) + c_2 e^{-3t} \sin(5t).$$

The particular solution is of the form $y_p = Ae^{-t}$. Inserting it into the equation we obtain

$$A - 6A + 34A = 2 \quad \Longrightarrow \quad 29A = 2 \quad \Longrightarrow \quad A = \frac{2}{29}.$$

Thus the general solution is

$$y(t) = c_1 e^{-3t} \cos(5t) + c_2 e^{-3t} \sin(5t) + \frac{2}{29} e^{-t}$$

and as a result

$$0 = y(0) = c_1 + 0 + \frac{2}{29} \quad \Longrightarrow \quad c_1 = -\frac{2}{29}$$

Differentiating, we find

$$y'(t) = -3c_1 e^{-3t} \cos(5t) - 5c_1 e^{-3t} \sin(5t) - 3c_2 e^{-3t} \sin(5t) + 5c_2 e^{-3t} \cos(5t) - \frac{2}{29} e^{-t}$$

and as a result

$$0 = y'(0) = -3c_1 + 5c_2 - \frac{2}{29} \quad \Longrightarrow \quad 5c_2 = 3c_1 + \frac{2}{29} \quad \Longrightarrow \quad c_2 = -\frac{6}{145} + \frac{2}{145} = -\frac{4}{145}.$$

Thus the solution to initial value problem is

$$\boxed{y(t) = -\frac{2}{29} e^{-3t} \cos(5t) - \frac{4}{145} e^{-3t} \sin(5t) + \frac{2}{29} e^{-t}.}$$

(b) Describe the long term behavior of the mass.
Solution. Taking a limit as $t \to \infty$ of the solutin above we find

$$\boxed{\lim_{t \to \infty} y(t) = 0.}$$

6. Find the general solution for the damped spring-mass problem

$$y'' + 4y = \sin(2t).$$

Solve with initial conditions $y(0) = 0$, $y'(0) = 1$.

Solution. First we consider the homogeneous problem

$$y'' + 4y = 0.$$

The corresponding characteristic equation is

$$r^2 + 4 = 0 \quad \Longrightarrow \quad r = \pm 2i.$$

The roots are pure imaginary, the general solution to homogeneous problem is

$$y_H = c_1 \cos(2t) + c_2 \sin(2t).$$

Since $\sin(2t)$ is part of the homogeneous solution the particular solution is of the form

$$y_p = At\cos(2t) + Bt\sin(2t).$$

Differentiating, we compute

$$y_p' = A\cos(2t) + B\sin(2t) - 2At\sin(2t) + 2Bt\cos(2t),$$

and

$$y_p'' = -2A\sin(2t) + 2B\cos(2t) - 2A\sin(2t) + 2B\cos(2t) - 4At\cos(2t) - 4Bt\sin(2t),$$

or

$$y_p'' = -4A\sin(2t) + 4B\cos(2t) - 4At\cos(2t) - 4Bt\sin(2t).$$

Thus y_p satisfies

$$-4A\sin(2t) + 4B\cos(2t) - 4At\cos(2t) - 4Bt\sin(2t) + 4At\cos(2t) + 4Bt\sin(2t) = \sin(2t),$$

or

$$-4A\sin(2t) + 4B\cos(2t) = \sin(2t).$$

Thus, $-4A = 1$ or $A = -\frac{1}{4}$ and $B = 0$. As a result the particular solution is

$$y_p = -\frac{t}{4}\cos(2t)$$

and the general solution is

$$y(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{t}{4}\cos(2t).$$

As a result

$$0 = y(0) = c_1.$$

Differentiating, we find (and using that $c_1 = 0$)

$$y'(t) = 2c_2\cos(2t) - \frac{1}{4}\cos(2t) + \frac{t}{2}\sin(2t)$$

and as a result

$$1 = y'(0) = 2c_2 - \frac{1}{4} \quad \Longrightarrow \quad 2c_2 = \frac{5}{4} \quad \Longrightarrow \quad c_2 = \frac{5}{8}.$$

Thus the solution to initial value problem is

$$\boxed{y(t) = \frac{5}{8}\sin(2t) - \frac{t}{4}\cos(2t).}$$