MATH 2410 – Differential Equations March 3, 2017

Practice Exam 2

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Consider the equation

$$y'(t) = (2-y)(1+y).$$

   (a) Solve with initial conditions $y(0) = 1$. **Solution:** Notice that the above equation is separable. Thus if $y \neq -1$ and $y \neq 2$, we have

$$\frac{dy}{(2-y)(1+y)} = dt.$$

   Since

$$\int \frac{dy}{(2-y)(1+y)} = \frac{1}{3}\int \left(\frac{1}{1+y} + \frac{1}{2-y}\right) dy = \frac{1}{3}\ln\left|\frac{1+y}{2-y}\right|,$$

   integrating both sides we find

$$\frac{1}{3}\ln\left|\frac{1+y}{2-y}\right| = t + C,$$

   or

$$\frac{1+y}{2-y} = Ce^{3t}, \quad \text{for any } C > 0.$$

   Using the initial condition we find

$$\frac{1+1}{2-1} = C, \quad \text{or} \quad C = 2.$$

   Thus

$$\frac{1+y}{2-y} = 2e^{3t} \quad \text{and as a result} \quad 1+y = 2e^{3t}(2-y).$$

   Solving for y we find

$$y = \frac{4e^{3t}-1}{2e^{3t}+1}.$$

   (b) What is the long time behaviour of the solution with $y(0) = 1$, i.e. compute $\lim_{t\to\infty} y(t)$. **Solution:**

   We need to compute $\lim_{t\to\infty} y(t)$. Diving by e^{3t}, we find

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{4e^{3t}-1}{2e^{3t}+1} = \lim_{t\to\infty} \frac{4-e^{-3t}}{2+e^{-3t}} = 2.$$

   (c) Confirm your answer by sketching the slope field of the equation.

   Solution: Will discuss in class. But notice that $y = 2$ and $y = -1$ are both (constant) solutions to the equation and $y' > 0$ for $y \in (-1, 2)$.

2. (20 points) Find the general solution for the problem

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = x + 2y.$$

Solve with initial conditions $x(0) = 0$, $y(0) = 1$.

Solution:

The general solution to the first equation is $x(t) = Ce^t$. Because of the initial condition $x(0) = 0$, we find that $x(t) = 0$. Thus the second equation reduces to $\frac{dy}{dt} = 2y$, which has a general soution $y(t) = Ce^{2t}$. Using the initial condition $y(0) = 1$, we find that $C = 0$ and $y(t) = e^{2t}$. Thus a pair $x(t) = 0$ and $y(t) = e^{2t}$ is the solution to the above system.

3. (20 points) The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(2 - x - y)$$
$$\frac{dy}{dt} = y(3 - x - y).$$

Draw the curves $x(t)$ and $y(t)$ if $x(0) = 1$, $y(0) = 1$ and $x(0) = 10$, $y(0) = 1$ in the phase plane.

Solution:

Will discuss in class.

4. (20 points)

A person opens a savings account with an initial deposit of \$10,000 and subsequently deposits \$100 each month. Find the value of the account at time $t > 0$, assuming that the bank pays 1% interest compounded continuously.

Solution:

Let $Q(t)$ denotes the value of the account at time t, measured in years. Thus, $Q(0) = 10000$. We make the simplifying assumption that the deposits are made continuously at a rate of \$1200 per year (\$100 each month). Thus the equation is

$$Q'(t) = 1200 + 0.01Q(t), \quad Q(0) = 10000.$$

The general solution to the homogeneous solution is $Q_H(t) = Ce^{0.01t}$. Thus $Q(t) = Q_H(t) + Q_P(t)$, where Q_P is any particular solution. Looking at the equation we expect the particular solutino to be a constant, call it A. Thus we expect $Q(t) = Ce^{\frac{t}{100}} + A$. Inserting it into the euqation we find

$$\frac{C}{100}e^{\frac{t}{100}} = 1200 + \frac{C}{100}e^{\frac{t}{100}} + \frac{A}{100}.$$

Thus, $0 = 1200 + \frac{A}{100}$ or $A = -1,200,000$. Using the initial condition we find

$$10,000 = Q(0) = C - 1,200,000, \quad \text{or} \quad C = 1,210,000.$$

Hence

$$Q(t) = 1,210,000e^{\frac{t}{100}} - 1,200,000.$$

5. (20 points) Compute the Euler's approximate solution at time $t = 1$ of the following equation

$$y'(t) = y(2 - y).$$

With initial position $y(0) = 1$ and time step $\Delta t = 0.5$

Solution:

The Euler's method for this problem is

$$y^{n+1} = y^n + \Delta t \cdot y^n \cdot (2 - y^n), \quad n = 0, 1, \ldots.$$

Since $\Delta t = 0.5$, in order to approximate $y(1)$, we only need two steps of the method. Since $y^0 = 1$, we find

$$y^1 = y^0 + \Delta t \cdot y^0 \cdot (2 - y^0) = 1 + 0.5 \cdot 1 \cdot (2 - 1) = \frac{3}{2}$$

and

$$y^2 = y^1 + \Delta t y^1 (2 - y^1) = \frac{3}{2} + \frac{1}{2} \cdot \frac{3}{2} \cdot (2 - \frac{3}{2}) = \frac{3}{2} + \frac{3}{8} = \frac{15}{8}.$$

Thus $y(1) \approx y^2 = \frac{15}{8}$.