Practice Exam 2

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Find the general solution for the problem

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = x + 2y.$$

Solve with initial conditions $x(0) = 1$, $y(0) = 3$.

Solution:

The system is decoupled. From the first equation we have that

$$x(t) = c_1 e^t, \quad c_1 \in \mathbb{R}.$$

Thus the second equation takes the form

$$\frac{dy}{dt} = c_1 e^t + 2y.$$

This is a linear equation and has the form

$$y = y_H + y_p,$$

where y_H is the general solution to a homogeneous equation

$$\frac{dy}{dt} = 2y$$

and y_p is any particular solution to the original equation. Thus

$$y_H = c_2 e^{2t}, \quad c_2 \in \mathbb{R}$$

and $y_p = Ae^t$ for some A we need to find out. Inserting it into the equation we have

$$Ae^t = 2Ae^t + c_1 e^t \quad \Rightarrow \quad A = -c_1.$$

Hence the general solution of the second equation is

$$y(t) = c_2 e^{2t} - c_1 e^t.$$

From the initial condition $x(0) = 1$, we find that $c_1 = 1$ and from $y(0) = 3$ that $c_2 = 4$. Thus the solution to the above initial value problem is

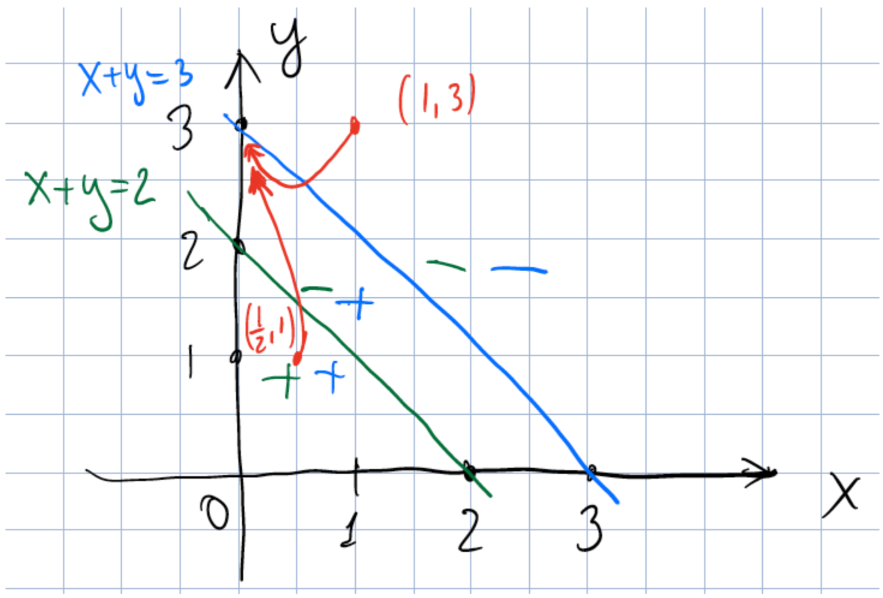
$$\boxed{x(t) = e^t, \quad y(t) = 4e^{2t} - e^t.}$$

2. (20 points) The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(2 - x - y)$$
$$\frac{dy}{dt} = y(3 - x - y).$$

Draw the curves $x(t)$ and $y(t)$ if $x(0) = 0.5$, $y(0) = 1$ and $x(0) = 1$, $y(0) = 3$ in the phase plane.

Solution: see the sketch. In summary no matter where you start the solution approaches the equilibrium solution $x = 0$ and $y = 3$.



3. (20 points) Consider the linear system $\vec{Y}' = A\vec{Y}$ where $\vec{Y} = (x(t), y(t))^T$

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 7 \end{pmatrix}$$

Find the general solution. Sketch the solution curves in the phase plane.

Solution: The characteristic polynomial is

$$det \begin{pmatrix} 4-\lambda & -2 \\ 1 & 7-\lambda \end{pmatrix} = (4-\lambda)(7-\lambda) + 2 = \lambda^2 - 11\lambda + 30 = (\lambda - 5)(\lambda - 6).$$

Hence the matrix A has two real eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 6$. Thus in order to use straight line solution method we need to find the corresponding eigenvectors.

For $\lambda_1 = 5$ we have

$$\begin{pmatrix} 4-5 & -2 \\ 1 & 7-5 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

For $\lambda_2 = 6$ we have

$$\begin{pmatrix} 4-6 & -2 \\ 1 & 7-6 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence the straight line solution is

$$\boxed{\vec{Y}(t) = c_1 e^{5t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.}$$

4. (20 points) Consider the linear system $\vec{Y}' = A\vec{Y}$ where $\vec{Y} = (x(t), y(t))^T$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Find the general solution. Solve for $x(0) = 1$, $y(0) = 2$.

Solution: The characteristic polynomial is

$$det \begin{pmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 + 1.$$

The roots are

$$(2-\lambda)^2 = -1 \quad \Rightarrow \quad 2-\lambda = \pm i \quad \Rightarrow \quad \lambda = 2 \pm i.$$

Hence the matrix A has two complex eigenvalues $\lambda_1 = 2+i$ and $\lambda_2 = 2-i$. Thus in order to find the solution we need just one eigenvalue, say $\lambda_1 = 2+i$ and the corresponding eigenvector.
For $\lambda = 2+i$ we have

$$\begin{pmatrix} 2-(2+i) & -1 \\ 1 & 2-(2+i) \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Hence using the Euler formula

$$\vec{Y}(t) = Ce^{(2+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = Ce^{2t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ -i \end{pmatrix} = Ce^{2t} \left[\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \right]$$

Since both real and complex parts are the solution, the general solution is

$$\vec{Y}(t) = C_1 e^{2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

To obtain the solution to initial value $x(0) = 1$, $y(0) = 2$, i.e. $\vec{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$., from the solution for $t = 0$ we have

$$\vec{Y}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Which give us $C_1 = 1$ and $C_2 = -2$. Thus, the solution to initial value problem is

$$\boxed{\vec{Y}(t) = e^{2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} - 2e^{2t} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.}$$

5. (20 points) A 200-gallon tank initially contains 2 pounds of sugar. Suppose water containing 0.5 pounds of sugar per gallon flows through one pipe into the tank at a rate of 5 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at rate 10 gallons per minute into a second 300-gallon tank that initially has no sugar. The water in the second tank is kept well mixed and well-mixed solution leaves the bottom of the tank at rate 10 gallons per minute.

Make a sketch of the problem and set up the initial value problem for the amount of sugar in the both tanks at time t (do not solve it).

Solution. Let $C_1(t)$ denote the amount of sugar (in pounds) in the first tank at time t. Thus $C_1(0) = 2$ and

$$\frac{dC_1}{dt} = C_1^{in} - C_1^{out}.$$

The amount of sugar that gets in $C_1^{in} = 5\ gal/min * 0.5\ lb/gal = 2.5\ lb/min$. The volume of the first tank varies in time, and decreases at the rate of 5 gallons per minute. The amount of sugar that gets out $C_1^{out} = C_1(t)/(200-5t) lb/gal * 10\ gal/min = 10C_1(t)/(200-5t)\ lb/min$. Thus the differential equation for the first tank is

$$\frac{dC_1}{dt} = 2.5 - \frac{10C_1(t)}{200-5t}.$$

The situation for the second tank is similar. Let $C_2(t)$ denote the amount of sugar (in pounds) in the second tank at time t. Thus $C_2(0) = 0$ and

$$\frac{dC_2}{dt} = C_2^{in} - C_2^{out}.$$

The amount of sugar that gets in

$$C_2^{in} = 10\ gal/min * C_1(t)/(200-5t)\ lb/gal.$$

The volume of the second tank does not change in time and stays at 300 gallons. The amount of sugar that gets out $C_2^{out} = C_2(t)/300 lb/gal * 10\ gal/min = C_2(t)/30\ lb/min$. Thus the differential equation for the second tank is

$$\frac{dC_2}{dt} = \frac{10C_1(t)}{200-5t} - \frac{C_2(t)}{30}.$$

Thus the initial value problem is

$$\boxed{\begin{array}{l} \dfrac{dC_1}{dt} = 2.5 - \dfrac{10C_1(t)}{200-5t} \\ \dfrac{dC_2}{dt} = \dfrac{10C_1(t)}{200-5t} - \dfrac{C_2(t)}{30} \\ C_1(0) = 2, \quad C_2(0) = 0. \end{array}}$$

The above system is actually decoupled and can be easily solved.