MATH 2410 – Differential Equations March 22, 2018

Practice Exam 2

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Find the general solution for the system

$$y'' - 5y' + 4y = 1 + t$$

Solve with initial conditions $y(0) = 1$, $y'(0) = 0$.

Solution: Since the equation is linear the solution is of the form

$$y = y_H + y_p,$$

where y_H is the general solution to a homogeneous equation

$$y'' - 5y' + 4y = 0$$

and y_p is any particular solution to the original equation. We seek the general solution of the homogeneous equation in the form e^{st} for some $s \in \mathbb{R}$. Inserting it into the equation we obtain

$$s^2 e^{st} - 5se^{st} + 4e^{st} = 0$$

or

$$s^2 - 5s + 4 = (s-4)(s-1) = 0.$$

Thus we have two solutions $s = 4$ and $s = 1$. Hence the general solution to the homogeneous problem is

$$y_H = c_1 e^{4t} + c_2 e^t, \quad c_1, c_2 \in \mathbb{R}.$$

The particular solution must be of the form $y_p = A + Bt$ for some $A, B \in \mathbb{R}$. Inserting it into the original equation we obtain

$$-5B + 4A + 4Bt = 1 + t.$$

Thus $4B = 1$ or $B = \frac{1}{4}$ and $-\frac{5}{4} + 4A = 1$ or $4A = \frac{9}{4}$, hence $A = \frac{9}{16}$. As a result the general form of the solution is

$$y(t) = c_1 e^{4t} + c_2 e^t + \frac{9}{16} + \frac{t}{4}.$$

Notice that

$$y'(t) = 4c_1 e^{4t} + c_2 e^t + \frac{1}{4}.$$

From $y(0) = 1$ and $y'(0) = 0$ we find

$$c_1 + c_2 + \frac{9}{16} = 1$$
$$4c_1 e + c_2 + \frac{1}{4} = 0$$

or

$$c_1 + c_2 = \frac{7}{16}$$
$$4c_1 e + c_2 = -\frac{1}{4}$$

Subtracting the first equation from the second one we obtain

$$3c_1 = -\frac{1}{4} - \frac{7}{16} = -\frac{11}{16} \quad \Rightarrow \quad c_1 = -\frac{11}{48}$$

and from the first equation we find

$$c_2 = \frac{7}{16} + \frac{11}{48} = \frac{32}{48}.$$

Thus the solution to the initial value problem is

$$\boxed{y(t) = -\frac{11}{48}e^{4t} + \frac{32}{48}e^t + \frac{9}{16} + \frac{t}{4}.}$$

2. (20 points) Find the general solution for the problem

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = -y.$$

Solve with initial conditions $x(0) = 1$, $y(0) = 2$.

Solution:

The system is decoupled. From the second equation we have that

$$y(t) = c_1 e^{-t}, \quad c_1 \in \mathbb{R}.$$

Thus the first equation takes the form

$$\frac{dx}{dt} = x + 2c_1 e^{-t}.$$

This is a linear equation and has the form

$$x = x_H + x_p,$$

where x_H is the general solution to a homogeneous equation

$$\frac{dx}{dt} = x$$

and x_p is any particular solution to the original equation. Thus

$$x_H = c_2 e^t, \quad c_2 \in \mathbb{R}$$

and $x_p = Ae^{-t}$ for some A we need to find out. Inserting it into the equation we have

$$-Ae^{-t} = Ae^{-t} + 2c_1 e^{-t} \quad \Rightarrow \quad A = -c_1.$$

Hence the general solution of the second equation is

$$x(t) = c_2 e^t - c_1 e^{-t}.$$

From the initial condition $y(0) = 2$, we find that $c_1 = 2$ and from $x(0) = 1$ that $c_2 = 2$. Thus the solution to the above initial value problem is

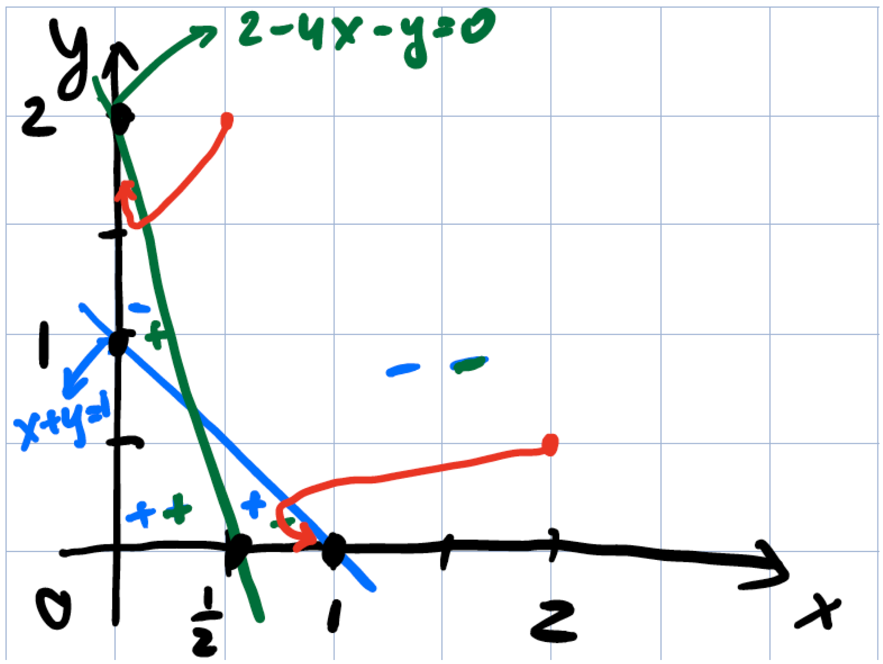
$$\boxed{y(t) = e^{-t}, \quad x(t) = 2e^t - e^{-t}.}$$

3. (20 points) The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(2 - 4x - y).$$

Draw the curves $x(t)$ and $y(t)$ if $x(0) = 0.5$, $y(0) = 2$ and $x(0) = 2$, $y(0) = 0.5$ in the phase plane.

Solution: see the sketch. In summary the solution to the problem is sensitive where you start. Thus if we start at $(0.5, 2)$ the solution approaches the equilibrium solution $x = 0$, $y = 2$ i.e. the y species win. On the other hand if we start at $(2, 0.5)$ the solution approaches the equilibrium solution $x = 1$, $y = 0$ i.e. the x species win.



4. (20 points) Consider the linear system $\vec{Y}' = A\vec{Y}$ where $\vec{Y} = (x(t), y(t))^T$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Find the general solution. Solve for $x(0) = 1$, $y(0) = 2$.

Solution:

The characteristic polynomial is

$$det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

Hence the matrix A has two real eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$. Thus in order to use straight line solution method we need to find the corresponding eigenvectors.

For $\lambda_1 = 3$ we have

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 1$ we have

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence the straight line solution is

$$\vec{Y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

To obtain the solution to initial value $x(0) = 1$, $y(0) = 2$, i.e. $\vec{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. From the solution for $t = 0$ we have

$$\vec{Y}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Which is equivalent to the system

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 2. \end{aligned}$$

Solving we obtain $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. Thus, the solution to initial value problem is

$$\boxed{\vec{Y}(t) = \frac{3}{2} e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.}$$

5. (20 points) Compute the Euler's approximate solution at time $t = 1$ of the following system

$$\begin{aligned} \frac{dx}{dt} &= xy + 2t \\ \frac{dy}{dt} &= y - x. \end{aligned}$$

With initial position $x(0) = 1$ and $y(0) = 0$ and time step $\Delta t = 0.5$

Solution: Let $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and the function $\vec{F}(\vec{Y}, t) = \vec{F}(x, y, t) = \begin{pmatrix} xy + 2t \\ y - x \end{pmatrix}$.

Given $\vec{Y}^0$, the backward Euler formula is

$$\vec{Y}^{n+1} = \vec{Y}^n + \Delta t \vec{F}(\vec{Y}^n, t_n), \quad n = 0, 1, 2, \ldots$$

In our problem $\vec{Y}^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Delta t = 0.5$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$ and we need to compute $\vec{Y}^2$, i.e. two steps of backward Euler method.

$$\vec{Y}^1 = \vec{Y}^0 + \frac{1}{2} \vec{F}(1, 0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 + 0 \\ 0 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}.$$

$$\vec{Y}^2 = \vec{Y}^1 + \frac{1}{2} \vec{F}(1, -0.5, 0.5) = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\frac{1}{2} + 2\frac{1}{2} \\ -\frac{1}{2} - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ -\frac{5}{4} \end{pmatrix}.$$

Thus the Euler approximation to the above system is

$$\boxed{x(1) = \frac{5}{4} \quad y(1) = -\frac{5}{4}.}$$