MATH 2410 – Differential Equations March 20, 2018

Practice Exam 2

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Find the general solution for the system

$$y'' + 6y' + 5y = e^t$$

Solve with initial conditions $y(0) = 0$, $y'(0) = 1$.

Solution: Since the equation is linear the solution is of the form

$$y = y_H + y_p,$$

where y_H is the general solution to a homogeneous equation

$$y'' + 6y' + 5y = 0$$

and y_p is any particular solution to the original equation. We seek the general solution of the homogeneous equation in the form e^{st} for some $s \in \mathbb{R}$. Inserting it into the equation we obtain

$$s^2 e^{st} + 6se^{st} + 5e^{st} = 0$$

or

$$s^2 + 6s + 5 = (s + 5)(s + 1) = 0.$$

Thus we have two solutions $s = -5$ and $s = -1$. Hence the general solution to the homogeneous problem is

$$y_H = c_1 e^{-5t} + c_2 e^{-t}, \quad c_1, c_2 \in \mathbb{R}.$$

Since the e^t is not part of the homogeneous solution, the particular solution must be of the form $y_p = Ae^t$. Inserting it into the original equation we obtain

$$Ae^t + 6Ae^t + 5Ae^t = e^t.$$

Thus $12A = 1$ and $A = \frac{1}{12}$. As a result the general form of the solution is

$$y(t) = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{12} e^t.$$

Notice that

$$y'(t) = -5c_1 e^{-5t} - c_2 e^{-t} + \frac{1}{12} e^t.$$

From $y(0) = 0$ and $y'(0) = 1$ we find

$$c_1 + c_2 = -\frac{1}{12}$$
$$-5c_1 e - c_2 = 1 - \frac{1}{12}.$$

Adding these two equations we obtain

$$-4c_1 = 1 - \frac{1}{6} = \frac{5}{6} \quad \Rightarrow \quad c_1 = -\frac{5}{24}$$

and from the first equation we find

$$c_2 = -\frac{1}{12} + \frac{5}{24} = \frac{3}{24} = \frac{1}{8}.$$

Thus the solution to the initial value problem is

$$\boxed{y(t) = -\frac{5}{24}e^{-5t} + \frac{1}{8}e^{-t} + \frac{1}{12}e^{t}.}$$

2. (20 points) Find the general solution for the problem

$$\begin{aligned}\frac{dx}{dt} &= x \\ \frac{dy}{dt} &= x + 2y.\end{aligned}$$

Solve with initial conditions $x(0) = 1$, $y(0) = 3$.

Solution:

The system is decoupled. From the first equation we have that

$$x(t) = c_1 e^t, \quad c_1 \in \mathbb{R}.$$

Thus the second equation takes the form

$$\frac{dy}{dt} = c_1 e^t + 2y.$$

This is a linear equation and has the form

$$y = y_H + y_p,$$

where y_H is the general solution to a homogeneous equation

$$\frac{dy}{dt} = 2y$$

and y_p is any particular solution to the original equation. Thus

$$y_H = c_2 e^{2t}, \quad c_2 \in \mathbb{R}$$

and $y_p = Ae^t$ for some A we need to find out. Inserting it into the equation we have

$$Ae^t = 2Ae^t + c_1 e^t \quad \Rightarrow \quad A = -c_1.$$

Hence the general solution of the second equation is

$$y(t) = c_2 e^{2t} - c_1 e^t.$$

From the initial condition $x(0) = 1$, we find that $c_1 = 1$ and from $y(0) = 3$ that $c_2 = 2$. Thus the solution to the above initial value problem is

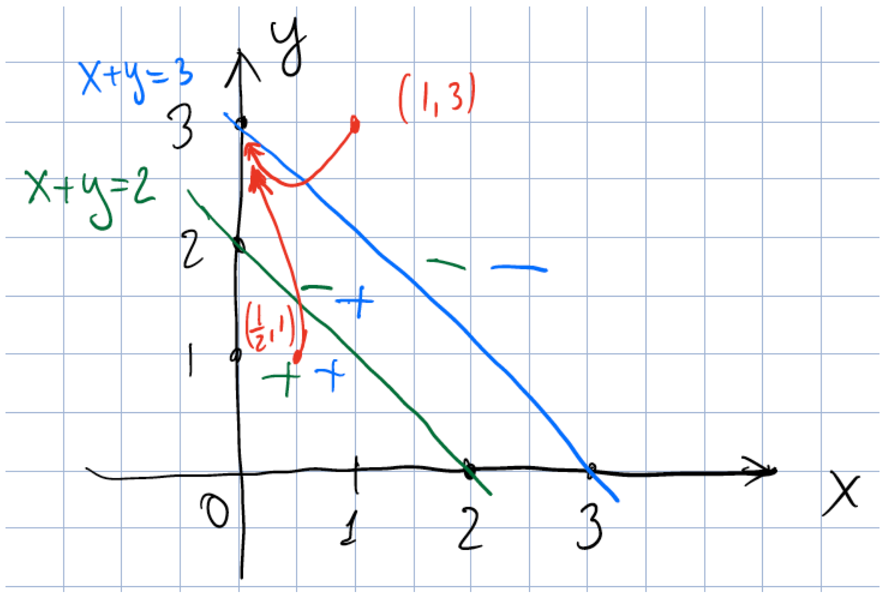
$$\boxed{x(t) = e^t, \quad y(t) = 2e^{2t} - e^t.}$$

3. (20 points) The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(2 - x - y)$$
$$\frac{dy}{dt} = y(3 - x - y).$$

Draw the curves $x(t)$ and $y(t)$ if $x(0) = 0.5$, $y(0) = 1$ and $x(0) = 1$, $y(0) = 3$ in the phase plane.

Solution: see the sketch. In summary no matter where you start the solution approaches the equilibrium solution $x = 0$ and $y = 3$.



4. (20 points) Consider the linear system $\vec{Y}' = A\vec{Y}$ where $\vec{Y} = (x(t), y(t))^T$

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 7 \end{pmatrix}$$

Find the general solution. Solve for $x(0) = 1$, $y(0) = -1$. **Solution:** The characteristic polynomial is

$$det \begin{pmatrix} 4-\lambda & -2 \\ 1 & 7-\lambda \end{pmatrix} = (4-\lambda)(7-\lambda) + 2 = \lambda^2 - 11\lambda + 30 = (\lambda - 5)(\lambda - 6).$$

Hence the matrix A has two real eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 6$. Thus in order to use straight line solution method we need to find the corresponding eigenvectors.

For $\lambda_1 = 5$ we have

$$\begin{pmatrix} 4-5 & -2 \\ 1 & 7-5 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

For $\lambda_2 = 6$ we have

$$\begin{pmatrix} 4-6 & -2 \\ 1 & 7-6 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}.$$

Thus the corresponding eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence the straight line solution is

$$\vec{Y}(t) = c_1 e^{5t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

To obtain the solution to initial value $x(0) = 1$, $y(0) = -1$, i.e. $\vec{Y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$., from the solution for $t = 0$ we have

$$\vec{Y}(0) = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Which give us $c_1 = 0$ and $c_2 = 1$. Thus, the solution to initial value problem is

$$\boxed{\vec{Y}(t) = e^{6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.}$$

5. (20 points) Compute the Euler's approximate solution at time $t = 1$ of the following system

$$\frac{dx}{dt} = x(2 - t - y)$$
$$\frac{dy}{dt} = y(1 - x + t).$$

With initial position $x(0) = 1$ and $y(0) = 1$ and time step $\Delta t = 0.5$ **Solution:** Let $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and the function $\vec{F}(\vec{Y}, t) = \vec{F}(x, y, t) = \begin{pmatrix} x(2-t-y) \\ y(1-x+t) \end{pmatrix}$.

Given $\vec{Y}^0$, the backward Euler formula is

$$\vec{Y}^{n+1} = \vec{Y}^n + \Delta t \vec{F}(\vec{Y}^n, t_n), \quad n = 0, 1, 2, \ldots$$

In our problem $\vec{Y}^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\Delta t = 0.5$, $t_0 = 0$, $t_1 = 0.5$ and we need to compute $\vec{Y}^2$, i.e. two steps of backward Euler method.

$$\vec{Y}^1 = \vec{Y}^0 + \frac{1}{2}\vec{F}(1,1,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1(2-0-1) \\ 1(1-1+0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}.$$

$$\vec{Y}^2 = \vec{Y}^1 + \frac{1}{2}\vec{F}(1.5, 1, 0.5) = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{3}{2}(2-\frac{1}{2}-1) \\ 1(1-\frac{3}{2}+\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{3}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 1\frac{3}{8} \\ 1 \end{pmatrix}.$$

Thus the Euler approximation to the above system is

$$\boxed{x(1) = 1\frac{3}{8} \quad y(1) = 1.}$$