MATH 2410 – Differential Equations February 4, 2017

# Practice Exam 1. Solutions.

*No calculators. Show your work. Clearly mark each answer.*

1. (20 points) Check if the given function $y$ is a solution of the initial value problem

   (a) $y = x^2(1 + \ln x)$

$$y''(x) = \frac{3xy'(x) - 4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e.$$

**Solution.** First we check the initial conditions.

$$y(e) = e^2(1 + \ln e) = 2e^2.$$

Using the product rule we find

$$y'(x) = 2x(1 + \ln x) + x \quad \text{and as a result} \quad y'(e) = 2e(1 + \ln e) + e = 5e.$$

Again using the product rule, we compute

$$y''(x) = 2 + 2\ln x + 2 + 1 = 5 + 2\ln x.$$

On the other hand

$$\frac{3xy'(x) - 4y}{x^2} = \frac{3x(2x(1 + \ln x) + x) - 4x^2(1 + \ln x)}{x^2} = 6 + 6\ln x + 3 - 4 - 4\ln x = 5 + 2\ln x.$$

That shows that $y = x^2(1 + \ln x)$ is a solution to

$$y''(x) = \frac{3xy'(x) - 4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e.$$

   (b) $y = \frac{x^2}{3} + x - 1$

$$y''(x) = \frac{x^2 - xy'(x) + y + 1}{x^2}, \quad y(1) = \frac{1}{3}, \quad y'(1) = \frac{5}{3}.$$

**Solution.**

Again, first we check the initial conditions.

$$y(1) = \frac{1^2}{3} + 1 - 1 = \frac{1}{3}.$$

Using the power rule we find

$$y'(x) = \frac{2x}{3} + 1 \quad \text{and as a result} \quad y'(1) = \frac{2}{3} + 1\frac{5}{3}.$$

Also we compute

$$y''(x) = \frac{2}{3}.$$

On the other hand

$$\frac{x^2 - xy'(x) + y + 1}{x^2} = \frac{x^2 - x(\frac{2x}{3} + 1) + \frac{x^2}{3} + x - 1 + 1}{x^2} = \frac{x^2 - \frac{2x^2}{3} + \frac{x^2}{3}}{x^2} = \frac{2}{3}.$$

That shows that $y = \frac{x^2}{3} + x - 1$ is a solution to

$$y''(x) = \frac{x^2 - xy'(x) + y + 1}{x^2}, \quad y(1) = \frac{1}{3}, \quad y'(1) = \frac{5}{3}.$$

2. (20 points) Find the general solution of

$$\frac{dy}{dt} + 2y = t^2 + 1.$$

**Solution 1.**
Computing the integrating factor

$$e^{\int 2dt} = e^{2t},$$

we have

$$e^{2t}\left(\frac{dy}{dt} + 2y\right) = e^{2t}(t^2+1)$$

which is the same as

$$\frac{d}{dt}\left(e^{2t}y\right) = e^{2t}(t^2+1).$$

Integrating both sides and using integrating by parts for $\int e^{2t}(t^2+1)\, dt$ twice, we find

$$e^{2t}y = \int e^{2t}(t^2+1)\, dt = \frac{e^{2t}}{2}(t^2+1) - \int e^{2t}t\, dt = \frac{e^{2t}}{2}(t^2+1) - \frac{e^{2t}}{2}t + \int \frac{e^{2t}}{2}\, dt = \frac{e^{2t}}{2}(t^2+1) - \frac{e^{2t}}{2}t + \frac{e^{2t}}{4} + C.$$

Diving by $e^{2t}$ we find

$$\boxed{y = \frac{1}{2}(t^2+1) - \frac{1}{2}t + \frac{1}{4} + Ce^{-2t} = \frac{2t^2-2t+3}{4} + Ce^{-2t}.}$$

**Solution 2.** The solution to homogeneous problem $y' + 2y = 0$ is

$$y_H(t) = ce^{-2t}.$$

Using the method of variation of parameter we assume that the solution $y$ is of the form

$$y(t) = c(t)e^{-2t}.$$

Plugging it into the equation $\frac{dy}{dt} + 2y = t^2 + 1$ we find

$$c'(t)e^{-2t} - 2c(t)e^{-2t} + 2c(t)e^{-2t} = t^2 + 1$$

which is the same as

$$c'(t)e^{-2t} = t^2 + 1.$$

Hence $c'(t) = e^{2t}(t^2+1)$ and integrating by parts twice as before we compute

$$c(t) = \int e^{2t}(t^2+1)\, dt = \frac{e^{2t}}{2}(t^2+1) - \int e^{2t}t\, dt = \frac{e^{2t}}{2}(t^2+1) - \frac{e^{2t}}{2}t + \int \frac{e^{2t}}{2}\, dt = \frac{e^{2t}}{2}(t^2+1) - \frac{e^{2t}}{2}t + \frac{e^{2t}}{4} + C$$

Hence

$$\boxed{y(t) = c(t)e^{-2t} = \frac{1}{2}(t^2+1) - \frac{1}{2}t + \frac{1}{4} + Ce^{-2t} = \frac{2t^2-2t+3}{4} + Ce^{-2t}.}$$

3. (20 points) Solve the initial value problem

$$\frac{dy}{dt} + \frac{2}{t}y = t^2$$
$$y(1) = 0.$$

**Solution.**

Computing the integrating factor

$$e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = e^{\ln|t|^2} = t^2,$$

we have

$$t^2\left(\frac{dy}{dt} + 2y\right) = t^4,$$

which is the same as

$$\frac{d}{dt}\left(t^2 y\right) = t^4.$$

Integrating both sides, we find

$$t^2 y = \int t^4 \; dt = \frac{t^5}{5} + C.$$

Diving by $t^2$ we find

$$y = \frac{t^3}{5} + \frac{C}{t^2}.$$

Using the initial condition we find

$$0 = y(1) = \frac{1}{5} + C \quad \Rightarrow \quad C = -\frac{1}{5}.$$

Thus, the solution initial boundary value problem is

$$\boxed{y = \frac{t^3}{5} - \frac{1}{5t^2}.}$$

4. (20 points) Consider the differential equation

$$\frac{dy}{dt} = yt^{\frac{2}{5}}.$$

(a) Compute the solution to the above differential equation.

(b) Is there a unique solution $y(t)$ to the above differential equation such that $y(0) = 0$? Why or why not?

(c) Is there a unique solution y(t) to the above differential equation such that $y(0) = 1$? Why or why not?

**Solution.**

(a) The equation is separable. Rewiritng the equation we have

$$\frac{dy}{y} = t^{\frac{2}{5}} dt,$$

provided $y \neq 0$. Integrating both sides we find

$$\ln|y| = \frac{5}{7}t^{\frac{7}{5}} + C, \quad C \in \mathbb{R}$$

and since $y = 0$ is also a solution

$$\boxed{y = Ce^{\frac{5}{7}t^{\frac{7}{5}}}, \quad C \geq 0.}$$

(b) Since the functions $f(t,y) = yt^{\frac{2}{5}}$ and $f_y(t,y) = t^{\frac{2}{5}}$ are continuous on any rectangle containing $(0,0)$, by the Existence and Uniquness Theorem there is a unique solution to the above differential equation with $y(0) = 0$.

(c) Again, since the functions $f(t,y) = yt^{\frac{2}{5}}$ and $f_y(t,y) = t^{\frac{2}{5}}$ are continuous on any rectangle containing $(0,1)$, by the Existence and Uniquness Theorem there is a unique solution to the above differential equation with $y(0) = 1$.

5. (20 points) A 400-gallon tank initially contains 200 gallons of sugar water at concentration of 0.1 pounds of sugar per gallon. Suppose water containing 0.5 sugar per gallon flows into the top of the tank at a rate of 2 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at rate 1 gallon per minute. How much sugar is in the tank when the tank is full?

**Solution.**

Let $C(t)$ denotes the amount of sugar at time $t$. Thus, we have $C(0) = 0.1\ lb./gal. \times 200\ gal. = 20\ lb.$ and we need to find $C(200)$. Let's look at $C'(t)$ i.e. at how the sugar in the tank changes. First of all there is a constant inflow of sugar at the rate of $0.5\ lb./gal. \times 2\ gal./min. = 1\ gal./min.$ and there is a outflow of $C(t)\ lb. \times 1\ gal./min. \div V(t)\ gal.$, where $V(t)$ is the volume of the water in the tank notice at time $t$, notice that it changes in time. Easy to see that $V(t) = 200 + t\ gal.$ Notice that all units do agree. Thus the equation for $C(t)$ is

$$C'(t) = 1 - \frac{C(t)}{200+t}, \quad C(0) = 20.$$

Rewriting it we have

$$C'(t) + \frac{C(t)}{200+t} = 1.$$

The integrating factor is

$$e^{\int \frac{dt}{200+t}} = e^{\ln|200+t|} = 200 + t,$$

Thus,

$$\frac{d}{dt}\left((200+t)C(t)\right) = 200 + t.$$

Integrating both sides we find

$$(200+t)C(t) = 200t + \frac{t^2}{2} + C,$$

or

$$C(t) = \frac{200t + \frac{t^2}{2} + C}{200+t}.$$

Using that $C(0) = 20$ we find

$$20 = \frac{C}{200} \quad \Rightarrow \quad C = 4000.$$

Hence

$$C(t) = \frac{200t + \frac{t^2}{2} + 4000}{200+t}.$$

And finally,

$$\boxed{C(200) = \frac{200 \cdot 200 + \frac{200 \cdot 200}{2} + 4000}{200+200} = \frac{200 + 100 + 20}{2} = 160\ lb.}$$