Practice Exam 1. Solutions.

1. (15 points) Consider the autonomous differential equation

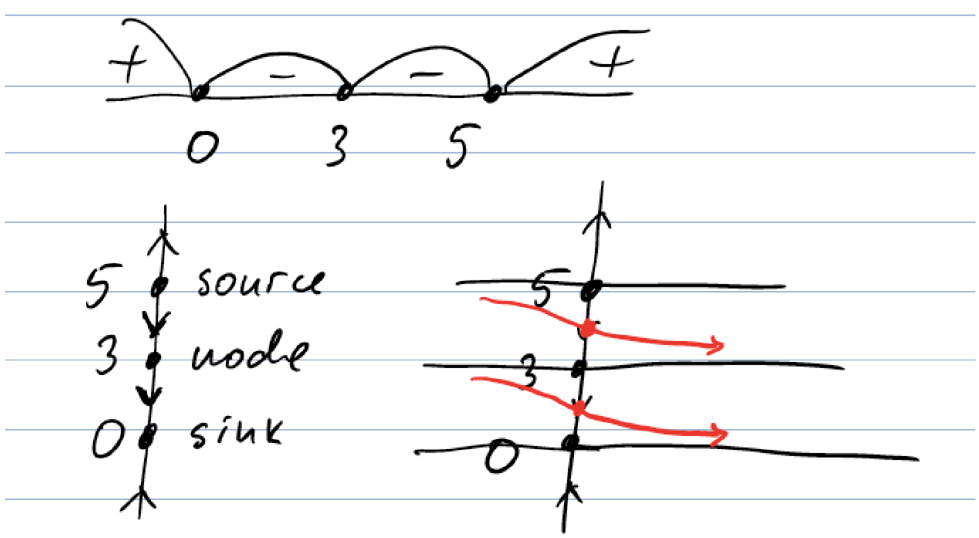
$$\frac{dy}{dt} = y(y-3)^2(y-5).$$

   (a) Compute the equilibrium solutions.
   (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.
   (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 4$ and $y(0) = 1$.

Solution.

(a) The equation is autonomous. The function $f(y) = y(y-3)^2(y-5)$ has 3 roots, namely $0, 3, 5$, hence the equilibrium solutions are $y = 0$, $y = 3$, and $y = 5$.

(b)



(c)

2. (15 points) Consider the linear equation

$$2y' + y = e^x$$

(a) Find the 1-parameter family of solution of the differential equation.

(b) Find the solution of the differential equation with the given initial value $y(0) = \alpha$.

(c) For what value(s) of α, the solution you found in (b) remains finite as $x \to \infty$?

Solution.

(a) The equation is linear, once we rewrite it in the form

$$y' + \frac{y}{2} = \frac{e^x}{2}$$

we see that the integrating factor is

$$\mu = e^{\int \frac{dx}{2}} = e^{\frac{x}{2}}.$$

Multiplying both sides by μ, we obtain

$$\frac{d}{dx}\left(ye^{\frac{x}{2}}\right) = \frac{e^{\frac{3x}{2}}}{2}.$$

Integrating both sides we obtain

$$ye^{x/2} = \int \frac{e^{3x/2}}{2} dx = \frac{e^{3x/2}}{3} + C,$$

hence

$$\boxed{y(x) = \frac{e^x}{3} + Ce^{-\frac{x}{2}}.}$$

(b) Taking $x = 0$ in the above expression we obtain

$$\alpha = y(0) = \frac{1}{3} + C \quad \Longrightarrow \quad C = \alpha - \frac{1}{3}$$

and as a result

$$y(x) = \frac{e^x}{3} + \left(\alpha - \frac{1}{3}\right) e^{-\frac{x}{2}}$$

(c)

3. (15 points) Consider the equation.

$$\frac{dy}{dt} = 2y(t^2 + 1).$$

(a) Find the general solution of the above equation

(b) Using the Euler method approximate $y(1)$ with initial condition $y(0) = 1$ and the time step $\Delta t = 0.5$.

Solution.

(a) The equation is separable. In addition $y = 0$ is a solution. For $y \neq 0$, we have

$$\frac{dy}{y} = 2(t^2 + 1)dt.$$

Integrating, we find

$$\ln|y| = \frac{2}{3}t^3 + 2t + C, \quad C \in \mathbb{R}$$

or

$$\boxed{y = C_1 e^{\frac{2}{3}t^3 + 2t}, \quad C_1 > 0.}$$

(b) The Euler method is

$$y^{n+1} = y^n + \Delta t f(t_n, y^n),$$

In this problem $f(t_n, y^n) = 2y^n(t_n^2 + 1)$, $y^0 = 1$, $\Delta t = 0.5$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$ and as a result we need to compute y^2.

$$y^1 = y^0 + \Delta t 2y^0(t_0^2 + 1) = 1 + 0.5 * 2 * 1 * (0 + 1) = 2,$$

$$y^2 = y^1 + \Delta t 2y^1(t_1^2 + 1) = 2 + 0.5 * 2 * 2(0.25 + 1) = 2 + 2.5 = 4.5.$$

Thus, we compute that with two time steps the approximate value of $y(1)$ is 4.5.

4. (15 points) Consider the following differential equation

$$(x^2y + y)\frac{dy}{dx} = -(xy^2 + x^2)$$

(a) Show that the above equation is exact

(b) Find the 1-parameter family of solution of the differential equation in implicit form.

(c) Find the particular solution to the initial value problem $y(0) = 2$.

Solution.

(a) Rewriting equation in the form

$$(xy^2 + x^2)dx + (x^2y + y)dy = 0$$

We see that the above equation is in the form

$$M(x, y)dx + N(x, y)dy = 0,$$

where

$$M(x, y) = (xy^2 + x^2) \quad N(x, y) = (x^2y + y).$$

Since

$$\frac{d}{dy}(xy^2 + x^2) = 2xy = \frac{d}{dx}(x^2y + y)$$

The equation is exact.

(b) Integrating $(xy^2 + x^2)$ in x, we find

$$\int (xy^2 + x^2)dx = \frac{1}{2}x^2y^2 + \frac{x^3}{3} + g(y).$$

Differentiating the above expression in y we obtain

$$\frac{d}{dy}\left(\frac{1}{2}x^2y^2 + \frac{x^3}{3} + g(y)\right) = x^2y + g'(y)$$

Comparing it to $N(x, y)$ we find that $g'(y) = y$ or $g(y) = \frac{y^2}{2}$. Hence the 1-parameter family of solution is

$$\boxed{\frac{1}{2}x^2y^2 + \frac{x^3}{3} + \frac{y^2}{2} = C.}$$

(c) From $y(0) = 2$ we find $2 = C$.

5. (15 points) Solve the following Bernoulli equation

$$xy' + y + x^2y^2e^x = 0.$$

by using the substitution $u = y^{-1}$.

Solution. Putting $y = -u^{-1}$, we find that

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}.$$

Substituting it into the equation we obtain the following equation for u

$$-\frac{x}{u^2}u' + \frac{1}{u} + \frac{x^2}{u^2}e^x = 0.$$

Multiplying it by u^2 and dividing it by x, we obtain

$$u' - \frac{u}{x} = xe^x,$$

which is a linear equation for u. The integrating factor is

$$e^{\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x},$$

and as a result

$$\frac{d}{dx}\left(\frac{u}{x}\right) = e^x.$$

Integrating, we obtain

$$\frac{u}{x} = e^x + C \quad \Longrightarrow \quad u = x(e^x + C)$$

and since $y = u^{-1}$, we obtain

$$\boxed{y(x) = \frac{1}{x(e^x + C)}.}$$

6. (20 points) A 400-gallon tank initially contains 1 pound of sugar. Suppose water containing 0.5 pounds of sugar per gallon flows into the top of the tank at a rate of 2 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at the same rate, 2 gallons per minute. How much sugar will be in the tank after 10 minutes? What does the concentration approach in the long run?

Solution. Let $C(t)$ denote the amount of sugar (in pounds) in the tank at time t. Thus $C(0) = 1$ and

$$\frac{dC}{dt} = C_{in} - C_{out}.$$

The amount of sugar that gets in $C_{in} = 2\ gal/min * 0.5\ lb/gal = 1\ lb/min$. The amount of sugar that gets out $C_{out} = C(t)/400 lb/gal * 2\ gal/min = C(t)/200\ lb/min$. Thus the differential equation is

$$\frac{dC}{dt} = 1 - \frac{C(t)}{200} = \frac{200 - C}{200}.$$

The equation is separable and for $C \neq 200$ we have

$$\frac{dC}{200 - C} = \frac{dt}{200}.$$

Integrating, we find

$$-\ln|200 - C| = \frac{t}{200} + C, \quad C \in \mathbb{R}$$

$$\ln|200 - C| = -\frac{t}{200} + C, \quad C \in \mathbb{R}$$

$$200 - C = C_1 e^{-\frac{t}{200}}, \quad C_1 > 0,$$

using the initial condition $C(0) = 1$, we find $C_1 = 199$, and as a result the amount of sugar at time t is

$$C(t) = 200 - 199e^{-\frac{t}{200}}.$$

After the 10 minutes the amount of sugar in the tank is

$$\boxed{C(10) = 200 - 199e^{-\frac{1}{20}} \ lb.}$$

Taking the limit as $t \to \infty$ we find that

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \left(200 - 199e^{-\frac{t}{200}}\right) = 200 \ lb,$$

hence the concentration in the long run approaches $\frac{200 \ lb}{400 \ gal} = 0.5 \ lb/gal$, the same concentration that the sugar flows in.