

Practice Exam 1. Solutions.

1. (15 points) Consider the autonomous differential equation

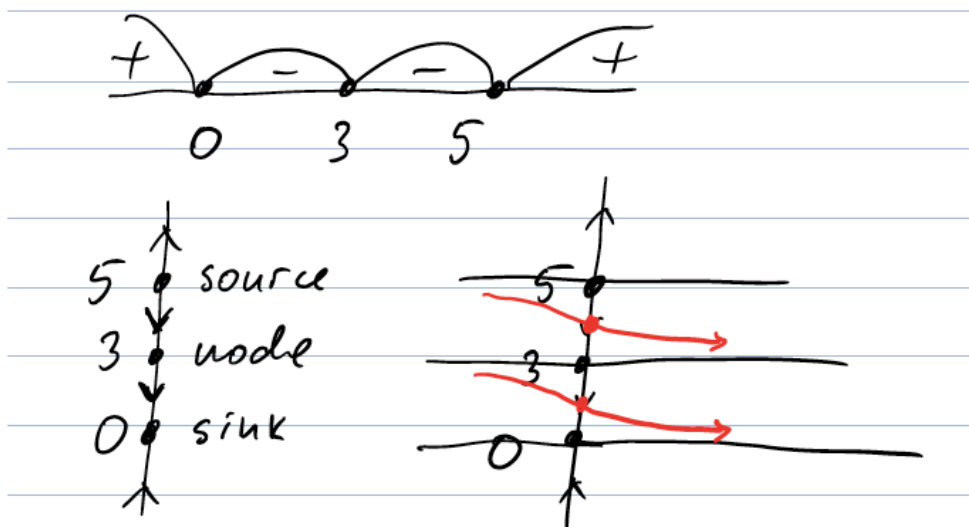
$$\frac{dy}{dt} = y(y - 3)^2(y - 5).$$

- (a) Compute the equilibrium solutions.
- (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.
- (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 4$ and $y(0) = 1$.

Solution.

(a) The equation is autonomous. The function $f(y) = y(y - 3)^2(y - 5)$ has 3 roots, namely 0, 3, 5, hence the equilibrium solutions are $y = 0$, $y = 3$, and $y = 5$.

(b)



(c)

2. (15 points) Consider the linear equation

$$2y' + y = e^x$$

- (a) Find the 1-parameter family of solution of the differential equation.
- (b) Find the solution of the differential equation with the given initial value $y(0) = \alpha$.
- (c) For what value(s) of α , the solution you found in (b) remains finite as $x \rightarrow \infty$?

Solution.

- (a) The equation is linear, once we rewrite it in the form

$$y' + \frac{y}{2} = \frac{e^x}{2}$$

we see that the integrating factor is

$$\mu = e^{\int \frac{dx}{2}} = e^{\frac{x}{2}}.$$

Multiplying both sides by μ , we obtain

$$\frac{d}{dx} (ye^{\frac{x}{2}}) = \frac{e^{\frac{3x}{2}}}{2}.$$

Integrating both sides we obtain

$$ye^{x/2} = \int \frac{e^{3x/2}}{2} dx = \frac{e^{3x/2}}{3} + C,$$

hence

$$y(x) = \frac{e^x}{3} + Ce^{-\frac{x}{2}}.$$

- (b) Taking $x = 0$ in the above expression we obtain

$$\alpha = y(0) = \frac{1}{3} + C \implies C = \alpha - \frac{1}{3}$$

and as a result

$$y(x) = \frac{e^x}{3} + \left(\alpha - \frac{1}{3}\right) e^{-\frac{x}{2}}$$

- (c)

3. (15 points) Consider the equation.

$$\frac{dy}{dt} = 2y(t^2 + 1).$$

- (a) Find the general solution of the above equation
- (b) Using the Euler method approximate $y(1)$ with initial condition $y(0) = 1$ and the time step $\Delta t = 0.5$.

Solution.

- (a) The equation is separable. In addition $y = 0$ is a solution. For $y \neq 0$, we have

$$\frac{dy}{y} = 2(t^2 + 1)dt.$$

Integrating, we find

$$\ln |y| = \frac{2}{3}t^3 + 2t + C, \quad C \in \mathbb{R}$$

or

$$y = C_1 e^{\frac{2}{3}t^3 + 2t}, \quad C_1 > 0.$$

(b) The Euler method is

$$y^{n+1} = y^n + \Delta t f(t_n, y^n),$$

In this problem $f(t_n, y^n) = 2y^n(t_n^2 + 1)$, $y^0 = 1$, $\Delta t = 0.5$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$ and as a result we need to compute y^2 .

$$y^1 = y^0 + \Delta t 2y^0(t_0^2 + 1) = 1 + 0.5 * 2 * 1 * (0 + 1) = 2,$$

$$y^2 = y^1 + \Delta t 2y^1(t_1^2 + 1) = 2 + 0.5 * 2 * 2(0.25 + 1) = 2 + 2.5 = 4.5.$$

Thus, we compute that with two time steps the approximate value of $y(1)$ is 4.5.

4. (15 points) Consider the following differential equation

$$(x^2 y + y) \frac{dy}{dx} = -(xy^2 + x^2)$$

(a) Show that the above equation is exact

(b) Find the 1-parameter family of solution of the differential equation in implicit form.

(c) Find the particular solution to the initial value problem $y(0) = 2$.

Solution.

(a) Rewriting equation in the form

$$(xy^2 + x^2)dx + (x^2 y + y)dy = 0$$

We see that the above equation is in the form

$$M(x, y)dx + N(x, y)dy = 0,$$

where

$$M(x, y) = (xy^2 + x^2) \quad N(x, y) = (x^2 y + y).$$

Since

$$\frac{d}{dy}(xy^2 + x^2) = 2xy = \frac{d}{dx}(x^2 y + y)$$

The equation is exact.

(b) Integrating $(xy^2 + x^2)$ in x , we find

$$\int (xy^2 + x^2)dx = \frac{1}{2}x^2 y^2 + \frac{x^3}{3} + g(y).$$

Differentiating the above expression in y we obtain

$$\frac{d}{dy} \left(\frac{1}{2}x^2 y^2 + \frac{x^3}{3} + g(y) \right) = x^2 y + g'(y)$$

Comparing it to $N(x, y)$ we find that $g'(y) = y$ or $g(y) = \frac{y^2}{2}$. Hence the 1-parameter family of solution is

$$\boxed{\frac{1}{2}x^2 y^2 + \frac{x^3}{3} + \frac{y^2}{2} = C.}$$

(c) From $y(0) = 2$ we find $2 = C$.

5. (15 points) Solve the following Bernoulli equation

$$xy' + y + x^2y^2e^x = 0.$$

by using the substitution $u = y^{-1}$.

Solution. Putting $y = -u^{-1}$, we find that

$$\frac{dy}{dx} = -u^{-2} \frac{du}{dx}.$$

Substituting it into the equation we obtain the following equation for u

$$-\frac{x}{u^2}u' + \frac{1}{u} + \frac{x^2}{u^2}e^x = 0.$$

Multiplying it by u^2 and dividing it by x , we obtain

$$u' - \frac{u}{x} = xe^x,$$

which is a linear equation for u . The integrating factor is

$$e^{\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x},$$

and as a result

$$\frac{d}{dx} \left(\frac{u}{x} \right) = e^x.$$

Integrating, we obtain

$$\frac{u}{x} = e^x + C \implies u = x(e^x + C)$$

and since $y = u^{-1}$, we obtain

$$\boxed{y(x) = \frac{1}{x(e^x + C)}}.$$

6. (20 points) A 400-gallon tank initially contains 1 pound of sugar. Suppose water containing 0.5 pounds of sugar per gallon flows into the top of the tank at a rate of 2 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at the same rate, 2 gallons per minute. How much sugar will be in the tank after 10 minutes? What does the concentration approach in the long run?

Solution. Let $C(t)$ denote the amount of sugar (in pounds) in the tank at time t . Thus $C(0) = 1$ and

$$\frac{dC}{dt} = C_{in} - C_{out}.$$

The amount of sugar that gets in $C_{in} = 2 \text{ gal/min} * 0.5 \text{ lb/gal} = 1 \text{ lb/min}$. The amount of sugar that gets out $C_{out} = C(t)/400 \text{ lb/gal} * 2 \text{ gal/min} = C(t)/200 \text{ lb/min}$. Thus the differential equation is

$$\frac{dC}{dt} = 1 - \frac{C(t)}{200} = \frac{200 - C}{200}.$$

The equation is separable and for $C \neq 200$ we have

$$\frac{dC}{200 - C} = \frac{dt}{200}.$$

Integrating, we find

$$-\ln |200 - C| = \frac{t}{200} + C, \quad C \in \mathbb{R}$$

$$\ln|200 - C| = -\frac{t}{200} + C, \quad C \in \mathbb{R}$$

$$200 - C = C_1 e^{-\frac{t}{200}}, \quad C_1 > 0,$$

using the initial condition $C(0) = 1$, we find $C_1 = 199$, and as a result the amount of sugar at time t is

$$C(t) = 200 - 199e^{-\frac{t}{200}}.$$

After the 10 minutes the amount of sugar in the tank is

$$\boxed{C(10) = 200 - 199e^{-\frac{1}{20}} \text{ lb.}}$$

Taking the limit as $t \rightarrow \infty$ we find that

$$\lim_{t \rightarrow \infty} C(t) = \lim_{t \rightarrow \infty} \left(200 - 199e^{-\frac{t}{200}} \right) = 200 \text{ lb},$$

hence the concentration in the long run approaches $\frac{200 \text{ lb}}{400 \text{ gal}} = 0.5 \text{ lb/gal}$, the same concentration that the sugar flows in.