Practice Exam 1. Solutions.

1. (20 points) Consider the autonomous differential equation

$$\frac{dy}{dt} = y(y-3)^2(y-5).$$

   (a) Compute the equilibrium solutions.

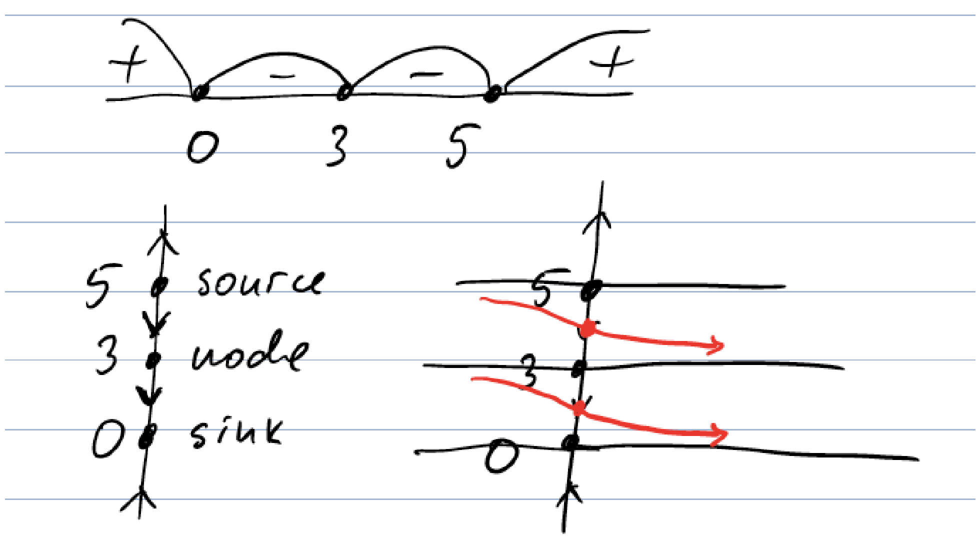
   (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.

   (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 4$ and $y(0) = 1$.

Solution.

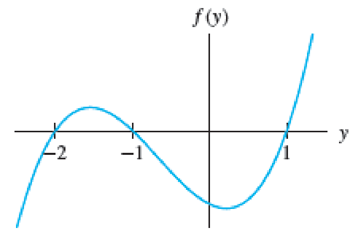
(a) The equation is autonomous. The function $f(y) = y(y-3)^2(y-5)$ has 3 roots, namely $0, 3, 5$, hence the equilibrium solutions are $y = 0$, $y = 3$, and $y = 5$.

(b)



(c)

2. (20 points)

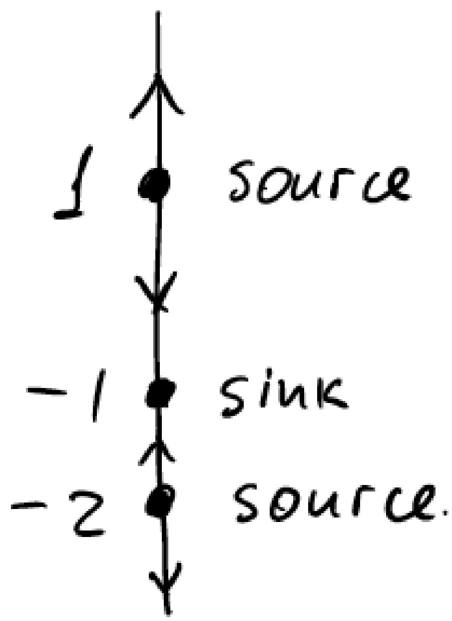


Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y),$$

with the graph of $f(y)$ on the right. Sketch the phase line and classify the equilibria as sinks, sources, or nodes.

Solution.



3. (20 points) Consider the equation.

$$\frac{dy}{dt} = 2y(t^2+1).$$

(a) Find the general solution of the above equation

(b) Using the Euler method approximate $y(1)$ with initial condition $y(0) = 1$ and the time step $\Delta t = 0.5$.

Solution.

(a) The equation is separable. In addition $y = 0$ is a solution. For $y \neq 0$, we have

$$\frac{dy}{y} = 2(t^2+1)dt.$$

Integrating, we find

$$\ln|y| = \frac{2}{3}t^3 + 2t + C, \quad C \in \mathbb{R}$$

or

$$y = C_1 e^{\frac{2}{3}t^3+2t}, \quad C_1 > 0.$$

(b) The Euler method is

$$y^{n+1} = y^n + \Delta t f(t_n, y^n),$$

In this problem $f(t_n, y^n) = 2y^n(t_n^2+1)$, $y^0 = 1$, $\Delta t = 0.5$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$ and as a result we need to compute y^2.

$$y^1 = y^0 + \Delta t 2y^0(t_0^2+1) = 1 + 0.5*2*1*(0+1) = 2,$$

$$y^2 = y^1 + \Delta t 2y^1(t_1^2+1) = 2 + 0.5*2*2(0.25+1) = 2 + 2.5 = 4.5.$$

Thus, we compute that with two time steps the approximate value of $y(1)$ is 4.5.

4. (20 points) Consider the differential equation

$$\frac{dy}{dt} = yt^{\frac{2}{5}}.$$

(a) Compute the general solution to the above differential equation.

(b) Is there a unique solution $y(t)$ to the above differential equation such that $y(0) = 0$? Why or why not?

(c) Is there a unique solution y(t) to the above differential equation such that $y(0) = 1$? Why or why not?

Solution.

(a) he equation is separable. In addition $y = 0$ is a solution. For $y \neq 0$, we have

$$\frac{dy}{y} = t^{\frac{2}{5}}dt.$$

Integrating, we find

$$\ln|y| = \frac{5}{7}t^{\frac{7}{5}} + C, \quad C \in \mathbb{R}$$

or

$$y = C_1 e^{\frac{5}{7}t^{\frac{7}{5}}}, \quad C_1 > 0.$$

(b) The function $f(t, y) = yt^{\frac{2}{5}}$ is continuous everywhere. In addition $\frac{\partial f}{\partial y} = t^{\frac{2}{5}}$ is continuous everywhere as well. So by the existence and uniqueness theorem for any initial condition there is a unique solution in the neighborhood of this point.

(c) Same as above.

5. (20 points) A 400-gallon tank initially contains 1 pound of sugar. Suppose water containing 0.5 pounds of sugar per gallon flows into the top of the tank at a rate of 2 gallons per minute. The water in the tank is kept well mixed and well-mixed solution leaves the bottom of the tank at the same rate, 2 gallons per minute. How much sugar will be in the tank after 10 minutes? What does the concentration approach in the long run?

Solution. Let $C(t)$ denote the amount of sugar (in pounds) in the tank at time t. Thus $C(0) = 1$ and

$$\frac{dC}{dt} = C_{in} - C_{out}.$$

The amount of sugar that gets in $C_{in} = 2\ gal/min * 0.5\ lb/gal = 1\ lb/min$. The amount of sugar that gets out $C_{out} = C(t)/400lb/gal * 2\ gal/min = C(t)/200\ lb/min$. Thus the differential equation is

$$\frac{dC}{dt} = 1 - \frac{C(t)}{200} = \frac{200 - C}{200}.$$

The equation is separable and for $C \neq 200$ we have

$$\frac{dC}{200 - C} = \frac{dt}{200}.$$

Integrating, we find

$$\begin{aligned} -\ln|200 - C| &= \frac{t}{200} + C, \quad C \in \mathbb{R} \\ \ln|200 - C| &= -\frac{t}{200} + C, \quad C \in \mathbb{R} \\ 200 - C &= C_1 e^{-\frac{t}{200}}, \quad C_1 > 0, \end{aligned}$$

using the initial condition $C(0) = 1$, we find $C_1 = 199$, and as a result the amount of sugar at time t is

$$C(t) = 200 - 199e^{-\frac{t}{200}}.$$

After the 10 minutes the amount of sugar in the tank is

$$C(10) = 200 - 199e^{-\frac{1}{20}}\ lb.$$

Taking the limit as $t \to \infty$ we find that

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \left(200 - 199e^{-\frac{t}{200}}\right) = 200\ lb,$$

hence the concentration in the long run approaches $\frac{200\ lb}{400\ gal} = 0.5\ lb/gal$, the same concentration that the sugar flows in.