MATH 2410 – Differential Equations April 28, 2016

Practice Final Exam

No calculators. Show your work. Clearly mark each answer.

1. Consider the autonomous differential equation

$$y' = (y+1)(y-3)^3(y-5)^2.$$

   (a) Compute the equilibrium solutions.

   (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.

   (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 2$.

2. A five gallon tank is full of water. We open a spigot so 1 gal. leaves the tank and introduce a mixture of 1/2 lb. per gal at 1 gal per minute. Assuming the mixture is well mixed, what is the concentration at time t?

3. Solve the initial value problem

$$y' - \frac{3y}{t+1} = (t+1)^2$$
$$y(0) = 3.$$

4. The following system describe a pair of competing species. Describe the long-time likely outcome of the competition by plotting the direction field.

$$\frac{dx}{dt} = x(2-x-y)$$
$$\frac{dy}{dt} = y(6-2x-2y).$$

   Draw the curves $x(t)$ and $y(t)$ if $x(0) = 3$ and $y(0) = 3$ in the phase plane.

5. Compute the Euler's approximate solution at time $t = 1$ of the following system

$$\frac{dx}{dt} = x(2-2x-y)$$
$$\frac{dy}{dt} = y(1-x-2y).$$

   With initial position $x(0) = 2$ and $y(0) = 1$ and time step $\Delta t = 0.5$

6. Consider the linear system $\vec{Y}' = A\vec{Y}$, where

$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -4 & -4 \\ -6 & -2 \end{pmatrix}$$

   (a) Compute the eigenvalues of A.

   (b) Classify the equilibrium at the origin (sink, spiral source, etc). Explain your answer.

   (c) What is the general solution to the system? Sketch the phase plane.

7. Compute the general solution to the linear system $\vec{Y}' = A\vec{Y}$, where

$$\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}$$

Sketch the phase plane.

8. Consider the spring-mass system whose motion is governed by

$$y'' + 6y' + 34y = 2e^{-t}.$$

   (a) Compute the solution to the above equation if $y(0) = 0$, $y'(0) = 0$.

   (b) Describe (in words) the long term behavior of the mass.

9. Find the general solution for the damped spring-mass problem

$$y'' + 4y = \sin(2t).$$

Solve with initial conditions $y(0) = 0$, $y'(0) = 1$.

10. Consider the equation

$$y' + 6y = e^{-2t}$$

with initial conditions $y(0) = 1$. Using the Laplace transform, find $y(t)$.

11. Consider the eqaution

$$y' + 9y = 1 + H_2(t)$$

with initial conditions $y(0) = 0$, where $H_2(t)$ is the Heavyside function,

$$H_2(t) = \begin{cases} 0, & 0 \le t < 2 \\ 1, & t \ge 2. \end{cases}$$

Using the Laplace transform, find $y(t)$.