Practice Final Exam. Solutions.

1. Find the standard matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:

Easy to see that the transformation T can be represented by a matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

2. True or False. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin though an angle $\pi/10$, then T is a linear transformation. Explain.

Solution: True, since the transformation T can be represented by a matrix

$$A = \begin{pmatrix} \cos\frac{\pi}{10} & \sin\frac{\pi}{10} \\ -\sin\frac{\pi}{10} & \cos\frac{\pi}{10} \end{pmatrix}.$$

3. Find the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}$$

Solution: First we find the characteristic polynomial and make it equal to zero,

$$det\begin{pmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{pmatrix} = (1-\lambda)det\begin{pmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{pmatrix} = (1-\lambda)[(5-\lambda)(-2-\lambda)-18] = 0$$

or

$$(1-\lambda)(\lambda-7)(\lambda+4) = 0.$$

Thus we have three distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 7$, and $\lambda_3 = -4$. To find the eigenvector corresponding to $\lambda_1 = 1$, we need to find a nontrivial solution to a homogeneous system

$$(A - I)\mathbf{v}_1 = \mathbf{0}$$

or

$$\begin{pmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ 6 & 7 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ 6 & 7 & -3 \end{pmatrix} \sim \begin{pmatrix} 4 & -2 & 3 \\ 0 & -20 & 15 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & -2 & 3 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

we have that x_3-free and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 - \frac{3}{4}x_3 \\ \frac{3}{4}x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{8}x_3 \\ \frac{3}{4}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{3}{8} \\ \frac{3}{4} \\ 1 \end{pmatrix} = 8x_3 \begin{pmatrix} -3 \\ 6 \\ 8 \end{pmatrix}.$$

Thus the first eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 6 \\ 8 \end{pmatrix}.$$

To find the second eigenvector corresponding to $\lambda_2 = 7$, we need to find a nontrivial solution to a homogeneous system

$$(A - 7I)\mathbf{v}_2 = \mathbf{0}$$

or

$$\begin{pmatrix} -2 & -2 & 3 \\ 0 & -6 & 0 \\ 6 & 7 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -2 & -2 & 3 \\ 0 & -6 & 0 \\ 6 & 7 & -9 \end{pmatrix} \sim \begin{pmatrix} -2 & -2 & 3 \\ 0 & -6 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

we have that $x_2 = 0$ and x_3-free and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}x_3 \\ 0 \\ x_3 \end{pmatrix} = 2x_3 \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

Thus the second eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

To find the third eigenvector corresponding to $\lambda_3 = -4$, we need to find a nontrivial solution to a homogeneous system

$$(A + 4I)\mathbf{v}_3 = \mathbf{0}$$

or

$$\begin{pmatrix} 9 & -2 & 3 \\ 0 & 5 & 0 \\ 6 & 7 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 9 & -2 & 3 \\ 0 & 5 & 0 \\ 6 & 7 & 2 \end{pmatrix} \sim \begin{pmatrix} 9 & -2 & 3 \\ 0 & 5 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

we have that $x_2 = 0$ and x_3-free and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} = 3x_3 \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$

Thus finally the third eigenvector is

$$\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix}$$

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Compute A^{10}.

Solution: First we find the characteristic polynomial

$$det(A - \lambda I) = det \begin{pmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{pmatrix} = (-2-\lambda)(5-\lambda) + 12 = (\lambda - 1)(\lambda - 2) = 0.$$

Thus we have two distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and as a result the corresponding eigenvalues are linearly independent and the matrix A is diagonalizable. To find the first eigenvector corresponding to $\lambda_1 = 1$, we need to find a nontrivial solution to a homogeneous system

$$(A - I)\mathbf{v}_1 = \mathbf{0}$$

or

$$\begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Easy to see that the solution is

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

To find the second eigenvector corresponding to $\lambda_2 = 2$, we need to find a nontrivial solution to a homogeneous system

$$(A - 2I)\mathbf{v}_2 = \mathbf{0}$$

or

$$\begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Easy to see that the solution is

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Thus,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$$

and

$$A = PDP^{-1}.$$

Thus,

$$A^{10} = PD^{10}P^{-1}.$$

which means

$$A^{10} = \begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix}^{10} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 4 - 3 \cdot 2^{10} & 12 \cdot 2^{10} - 12 \\ 1 - 2^{10} & 4 \cdot 2^{10} - 3 \end{pmatrix}.$$

5. For the following matrices A find the basis for $Nul(A)$, $Row(A)$, $Col(A)$. What is $rank(A)$?

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution: The matrix A is already in the echelon form. We can see that there are three nonzero rows, hence $rank(A) = 3$ and right the way we have

$$Row(A) = span\{\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$$

and

$$Col(A) = span\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}.$$

Thus the $dim(Nul(A)) = 2$ and to find the basis for $Nul(A)$ we need to find all nontrivial solutions to $A\mathbf{x} = \mathbf{0}$. From the matrix A we can see right the way that x_3, x_4-free and $x_5 = 0$. Thus,

$$2x_2 = -x_3 - x_4 \quad \text{and} \quad x_1 = -2x_2 - 2x_3 = -x_3 + x_4$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ -\frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence basis for $Nul(A)$

$$Nul(A) = span\{\begin{pmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}\}.$$

Notice that $Nul(A) \perp Row(A)$ i.e. any vector in the $Nul(A)$ is orthogonal to any vector in $Row(A)$.

6. If the null space of a 50×60 matrix A is 40-dimensional,

   (a) What is the rank of A?

   (b) $Null(A)$ is a subspace of $\mathbb{R}^n$, what is n?

   (c) $Col(A)$ is a subspace of $\mathbb{R}^n$, what is n?

   Solution:

   (a) $rank(A) = 60 - dim(Nul(A)) = 60 - 40 = 20$.

   (b) $Null(A)$ is a subspace of $\mathbb{R}^{60}$, since $A : \mathbb{R}^{60} \to \mathbb{R}^{50}$.

   (c) $Col(A)$ is a subspace of $\mathbb{R}^{50}$.

7. Let A be a n-by-n matrix that satisfies $A^2 = A$. What can you say about the determinant of A?

   Solution: Let $x = det(A)$. Since $det(AB) = det(A)det(B)$ the relation $A^2 = A$ implies $x^2 = x$ or $x^2 - x = x(x-1) = 0$. Hence $x = 0$ or $x = 1$. Thus we can conclude that the $det(A)$ is either equal to 1 or 0.

8. Suppose a 4×7 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^3$? Explain.

   Solution: Since A has four pivot columns, $rank(A) = 4$ and $dim(Nul(A)) = 3$. Since $Col(A)$ is a subset of $\mathbb{R}^4$ and is four dimensional we can conclude indeed that Col $A = \mathbb{R}^4$. However Nul A is a subset of $\mathbb{R}^7$.

9. Show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set in $\mathbb{R}^3$. Then express a vector $\mathbf{x}$ as a linear combination of $\mathbf{u}'s$, where

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

   Solution: The orthogonality we can check using a dot product. Thus,

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= 3 \cdot 2 - 3 \cdot 2 - 0 \cdot 1 = 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= 3 \cdot 1 - 3 \cdot 1 + 0 \cdot 4 = 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= 2 \cdot 1 + 2 \cdot 1 - 1 \cdot 4 = 0. \end{aligned}$$

   Nest we want to find coefficients $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

   Taking the dot product with $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ and using orthogonality we find

$$c_1 = \frac{\mathbf{u}_1 \cdot \mathbf{x}}{\mathbf{u}_1 \cdot \mathbf{u}_1}, \quad c_2 = \frac{\mathbf{u}_2 \cdot \mathbf{x}}{\mathbf{u}_2 \cdot \mathbf{u}_2}, \quad c_3 = \frac{\mathbf{u}_3 \cdot \mathbf{x}}{\mathbf{u}_3 \cdot \mathbf{u}_3}.$$

   Thus we find

$$\begin{aligned} c_1 &= \frac{\mathbf{u}_1 \cdot \mathbf{x}}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{3 \cdot 5 + 3 \cdot 3 + 0 \cdot 1}{3 \cdot 3 + 3 \cdot 3 + 0 \cdot 0} = \frac{24}{18} = \frac{4}{3}. \\ c_2 &= \frac{\mathbf{u}_2 \cdot \mathbf{x}}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{2 \cdot 5 - 2 \cdot 3 - 1 \cdot 1}{2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1} = \frac{3}{9} = \frac{1}{3}. \end{aligned}$$

   and

$$c_3 = \frac{\mathbf{u}_3 \cdot \mathbf{x}}{\mathbf{u}_3 \cdot \mathbf{u}_3} = \frac{1 \cdot 5 - 1 \cdot 3 + 4 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 4 \cdot 4} = \frac{6}{18} = \frac{1}{3}.$$

   Hence

$$\mathbf{x} = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3.$$

10. Using the Gram-Schmidt process to produce an orthogonal basis for $W = span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 7 \\ -7 \\ -4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Solution: Using the Gram-Schmidt we produce orthogonal vectors $\mathbf{x}_1$,$\mathbf{x}_2$, and $\mathbf{x}_3$ such that $W = span\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. First we take $\mathbf{x}_1 = \mathbf{v}_1$. Then

$$\mathbf{x}_2 = \mathbf{v}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{v}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 = \begin{pmatrix} 7 \\ -7 \\ -4 \\ 1 \end{pmatrix} - \frac{7+28+1}{1+16+1}\begin{pmatrix} 1 \\ -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7-2 \\ -7+8 \\ -4+0 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \\ -1 \end{pmatrix}.$$

Finally,

$$\mathbf{x}_3 = \mathbf{v}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{v}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{3}{18}\begin{pmatrix} 1 \\ -4 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{43}\begin{pmatrix} 5 \\ 1 \\ -4 \\ -1 \end{pmatrix}.$$