Practice Exam 2. Solutions.

1. Find A^{-1}, where
$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$
Using A^{-1} solve the system
$$\begin{aligned} 3x_1 + 2x_2 &= 2 \\ 4x_1 + 3x_2 &= 3. \end{aligned}$$

Solution.
Using that the inverse of 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $ad - bc \neq 0$, we find
$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}.$$
Since the system
$$\begin{aligned} 3x_1 + 2x_2 &= 2 \\ 4x_1 + 3x_2 &= 3. \end{aligned}$$
we can rewrite as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ we find $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$ or
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2. Find A^{-1}, where
$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$
Using A^{-1} solve the system
$$\begin{aligned} x_1 + 3x_2 &= 1 \\ x_2 + 3x_3 &= 2 \\ x_1 + x_3 &= 3. \end{aligned}$$

Solution.
Since matrix A is 3×3 matrix, to find A^{-1} we form an augmented matrix $(A|I)$ or
$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right).$$
Doing standard row operations we find
$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & -1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -10 & 1 & -3 & -1 \end{array} \right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -0.1 & 0.3 & 0.1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -0.3 & 0.1 & -0.3 \\ 0 & 0 & 1 & -0.1 & 0.3 & 0.1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.1 & -0.3 & 0.9 \\ 0 & 1 & 0 & -0.3 & 0.1 & -0.3 \\ 0 & 0 & 1 & -0.1 & 0.3 & 0.1 \end{array}\right)$$

Thus,

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 1 & -3 & 9 \\ -3 & 1 & -3 \\ -1 & 3 & 1 \end{pmatrix}.$$

Since the system

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ x_2 + 3x_3 &= 2 \\ x_1 + x_3 &= 3. \end{aligned}$$

we can rewrite as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ we find $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$ or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & -3 & 9 \\ -3 & 1 & -3 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.2 \\ -0.4 \\ 0.8 \end{pmatrix}.$$

3. Show that the following set S of polynomials is linearly independent and hence form a basis for $\mathbb{P}_2$

$$S = \{p_1(x), p_2(x), p_3(x)\},$$

where $p_1(x) = 1 - x$, $p_2(x) = 1 + x^2$, $p_3(x) = x + 2x^2$. Find the coordinates of the polynomial $p(x) = x^2 + x + 1$ in the basis S.

Solution.

First we will show that the set S forms a basis for $\mathbb{P}_2$. Since the dimension of $\mathbb{P}_2$ is 3 and there are 3 elements in the set S, we need only to show that the S is linearly independent. To show that assume there exist c_1, c_2, c_3 such that

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0$$

for all $x \in \mathbb{R}$. Thus using the definitions of $p_1(x), p_2(x), p_3(x)$ we have

$$c_1(1-x) + c_2(1+x^2) + c_3(x+2x^2) = 0 \quad \Longrightarrow (c_1+c_2) + x(c_3-c_1) + x^2(c_2+2c_3) = 0$$

for all $x \in \mathbb{R}$. Thus, we must have

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_3 - c_1 &= 0 \\ c_2 + 2c_3 &= 0. \end{aligned}$$

We can rewrite it in the matrix form

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix is non-singular (has a pivot in every column, no free variables), hence the homogeneous system has only the trivial solution. Thus, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. As a result the set S is linearly independent.

Now we need to find numbers a_1, a_2, a_3 such that

$$a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = x^2 + x + 1$$

for all $x \in \mathbb{R}$. Thus using the definitions of $p_1(x), p_2(x), p_3(x)$ we have

$$a_1(1-x) + a_2(1+x^2) + a_3(x+2x^2) = x^2+x+1 \quad \Longrightarrow (c_1+c_2) + x(c_3-c_1) + x^2(c_2+2c_3) = x^2+x+1$$

for all $x \in \mathbb{R}$. Thus, we must have

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_3 - a_1 &= 1 \\ a_2 + 2a_3 &= 1. \end{aligned}$$

We can rewrite it in the matrix form

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Forming an augmented matrix, we find

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 \\ -1 & 0 & 1 & | & 1 \\ 0 & 1 & 2 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 1 & 2 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$

Thus, $a_3 = -1$, $a_2 = 3$, and $a_1 = -2$. Thus the coordinates of $p(x) = x^2 + x + 1$ in the basis S are $-2, 3, -1$, i.e. $[p(x)]_S = (-2; 3; -1)$.

4. Show that T is a linear transformation by finding a matrix that implements the mapping. $T(x_1; x_2; x_3) = (2x_1 + 3x_2; 3x_1 + 2x_3; x_1 + x_2; x_2 + x_3)$.

   Solution.

   We can see that the mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ and the 3×4 matrix

$$\begin{pmatrix} 2 & 3 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

   implements this mapping.

5. For the following matrix A find the basis for $Nul(A)$, $Row(A)$, $Col(A)$, and compute the $rank(A)$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix}.$$

   Solution.

   Since

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & -3 & -4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -4 & 2 \\ 0 & 0 & 2 & -1 \end{pmatrix},$$

   we immediately find that the $rank(A) = 3$ and the basis for $row(A)$ are

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

the basis for $Col(A)$ are

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

To find the basis for $Nul(A)$ we need to find the solution set to

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -4 & 2 \\ 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus x_4 is free and $x_3 = \frac{1}{2}x_4$, $x_2 = -\frac{4}{3}x_3 + \frac{2}{3}x_4 = -\frac{2}{3}x_4 + \frac{2}{3}x_4 = 0$ and $x_1 = -2x_2 - 3x_3 = -\frac{3}{2}x_4$. Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -\frac{3}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} = 2x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Thus the basis for $Nul(A)$ is a vector $\begin{pmatrix} -3 \\ 0 \\ 1 \\ 2 \end{pmatrix}$.

6. Can a matrix A with a zero row have $Nul(A) = \{\mathbf{0}\}$? Explain.

   Solution.

   Yes, for example the matrix

   $$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   has a Null space $\mathbf{0}$. So is

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   On the other hand a square matrix can not.

7. If A is 100×99 matrix, what is the smallest possible dimension of the $Nul(A)$?

   Solution. Since for $m \times n$ matrix A, there is a relation $dim(Nul(A)) + rank(A) = n$, we have

   $$dim(Nul(A)) + rank(A) = 99.$$

   Since A is 100×99 matrix, $rank(A) \leq 99$, and the smallest possible dimension of the $Nul(A)$ is 0.

8. Is the following set a subspace of $\mathbb{R}^3$

   $$\left\{ \begin{pmatrix} r \\ s \\ t \end{pmatrix} : 2r = s + t \quad \text{and} \quad r = t - s \right\}?$$

   Explain.

   Solution.

   Since the relations we can write as

   $$2r - s - t = 0$$
   $$r - t + s = 0$$

we see that the subset is just the Null space of the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

which is a subspace of $\mathbb{R}^3$, since a Null space of a $m \times n$ matrix is a subspace of $\mathbb{R}^n$.