MATH 1132 – Calculus II May 2, 2019

Practice Final Exam. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. (15 points) Find the limits of the following sequences $\{a_n\}$.

(a)

$$a_n = \frac{n^2 - 2n + 1}{3n^3 + 1}$$

(b)

$$a_n = \frac{\sin n}{n^2}$$

(c)

$$a_n = \left(1 - \frac{1}{n}\right)^n$$

Solution:

(a) Since by the L'Hospital rule

$$\lim_{x\to\infty} \frac{x^2 - 2x + 1}{3x^3 + 1} = \lim_{x\to\infty} \frac{2x - 2}{9x^2} = \lim_{x\to\infty} \frac{2}{18x} = 0$$

it follows that

$$\lim_{n\to\infty} \frac{n^2 - 2n + 1}{3n^3 + 1} = 0.$$

(b) Since $-1 \leq \sin x \leq 1$ for any x, we have

$$-\frac{1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}.$$

Because $\lim_{n\to\infty} \frac{1}{n^2} = 0$ by the Sandwich theorem it follows that

$$\lim_{n\to\infty} \frac{\sin n}{n^2} = 0.$$

(c) Using that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

we find that

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{-n}\right)^{(-n)\cdot(-1)} = \left(\lim_{n\to\infty} \left(1 + \frac{1}{-n}\right)^{-n}\right)^{-1} = e^{-1} = \frac{1}{e}.$$

2. (20 points) Which of the following series converge or diverge? Give reasons to your answers.

(a)

$$\sum_{n=2}^{\infty} \frac{n^2 + 1}{(n^2 - 1)^3}$$

(b)

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{\ln n}}$$

(c)

$$\sum_{n=0}^{\infty} \frac{n}{n+1}$$

(d)

$$\sum_{n=0}^{\infty} \frac{1}{e^n}$$

Solution:

(a) Since terms of the series are positive we can use the Limit Comparison Test. The leading term in the numerator is n^2 and n^6 in the denominator we compare the original series to $\sum \frac{1}{n^4}$. Since

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{(n^2-1)^3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^4(n^2+1)}{(n^2-1)^3} = \lim_{n\to\infty} \frac{1+\frac{1}{n^4}}{(1-\frac{1}{n^2})^3} = 1 > 0$$

and we know that $\sum \frac{1}{n^4}$ converges as a p-series with $p = 4$, it follows that $\sum_{n=2}^{\infty} \frac{n^2+1}{(n^2-1)^3}$ converges as well.

(b) This is an alternating series. To show convergence we need to show that $\frac{1}{\sqrt{\ln n}}$ monotonically going to 0. Obviously

$$\lim_{n\to\infty} \frac{1}{\sqrt{\ln n}} = 0$$

and $\sqrt{\ln x}$ is monotonically increasing function. Thus by the Alternating Series Theorem $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{\ln n}}$ converges.

(c) Since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

by the necessary convergence theorem the series diverges.

(d) By the n-root test

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e} < 1$$

the series converges.

3. (20 points) Find the radii and the intervals of convergence of the following power series.

(a)

$$\sum_{n=1}^{\infty} \frac{4^n x^n}{n^3}$$

(b)

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$$

(c)

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{5^n n}$$

(d)

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{n!}$$

Solutions:

(a) By the n-Root Test we need to have

$$\lim_{n\to\infty} \sqrt[n]{\frac{4^n|x|^n}{n^3}} = 4|x| \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^3}} = 4|x| < 1,$$

or $|x| < 1/4$. Thus the radius of convergence is $1/4$. To find out the interval of convergence we need to check the convergence of the series for $x = \frac{1}{4}$ and $x = -\frac{1}{4}$.
For $x = \frac{1}{4}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

the series converges as p-series. For $x = -\frac{1}{4}$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} < \infty$$

by the Alternating Series Theorem. Thus, the interval of convergence is $[-\frac{1}{4}, \frac{1}{4}]$.

(b) By the n-Root Test we need to have

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x-3|^n}{2^n}} = \frac{|x-3|}{2} < 1,$$

or $|x-3| < 2$. Thus the radius of convergence is 2. To find out the interval of convergence we need to check the convergence of the series for $x = 5$ and $x = -1$.
For $x = 5$ we have

$$\sum_{n=1}^{\infty} 1 = \infty$$

the series diverges. For $x = -1$ we have

$$\sum_{n=1}^{\infty}(-1)^n$$

the series diverges since $\lim_{n\to\infty} a_n \neq 0$. Thus, the interval of convergence is $(-1, 5)$.

(c) By the n-Root Test we need to have

$$\lim_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{5^n} n} = \frac{x^2}{5} \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{x^2}{5} < 1,$$

or $x^2 < 5$ or $|x| < \sqrt{5}$. Thus the radius of convergence is $\sqrt{5}$. To find out the interval of convergence we need to check the convergence of the series for $|x| = \sqrt{5}$ or $x = \sqrt{5}$ and $x = -\sqrt{5}$.
For $x = \sqrt{5}$ we have

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

diverges as Harmonic series. Similarly for $x = -\sqrt{5}$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

the series converges as alternative series. Thus the interval of convergence is $(-\sqrt{5}, \sqrt{5})$.

(d) By the ratio test

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\frac{|x|^{2n+2}}{(n+1)!}}{\frac{|x|^{2n}}{(n)!}}=\lim_{n\to\infty}\frac{|x|^2}{(n+1)}=0<1$$

for all real x. Thus the radius of convergence is ∞ and the interval convergence is $(-\infty,\infty)$.

4. (15 points) Find the Maclaurin series of the following function

$$(x^2-1)e^{2x}.$$

Solutions: Using that the Maclaurin series for e^x

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

we find that

$$e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$$

and

$$\begin{aligned}(x^2-1)e^{2x}&=x^2e^{2x}-e^{2x}=\sum_{n=0}^{\infty}\frac{2^n x^{n+2}}{n!}-\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}\\&=\sum_{n=2}^{\infty}\frac{2^{n-2}x^n}{(n-2)!}-\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}\\&=-1-2x+\sum_{n=2}^{\infty}\left(\frac{2^{n-2}}{(n-2)!}-\frac{2^n}{n!}\right)x^n.\end{aligned}$$

5. (15 points) Find the quadratic ($n=2$) Taylor polynomial at $a=1$ of the following function

$$\sqrt[4]{x}.$$

Solution: Looking at the Taylor series formula

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n,$$

we can see that to derive the quadratic Taylor polynomial at $a=1$ we need to compute $f(1)$, $f'(1)$, and $f''(1)$. Thus,

$$\begin{aligned}f(1)&=1\\f'(x)=\frac{1}{4}x^{-3/4},\ \Rightarrow\ f'(1)&=\frac{1}{4}\\f''(x)=-\frac{3}{16}x^{-7/4},\ \Rightarrow\ f''(1)&=-\frac{3}{16},\end{aligned}$$

and the quadratic polynomial at $a=1$ is

$$T_2(x)=1+\frac{1}{4}(x-1)-\frac{3}{32}(x-1)^2.$$

6. (10 points) What is the largest d can be such that the approximation

$$\sin x\approx x-x^3/6$$

is accurate to 5 decimal places for $|x|\le d$?

Solution: We need find largest d such that the remainder $R_3(x) = \sin x - (x - x^3/6)$ is bounded by 10^{-5} for $|x| \le d$. Since the Maclaurin series for sine

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

is alternating, we have

$$|R_3(x)| \le \frac{|x|^5}{5!}.$$

Thus we want,

$$|R_3(x)| \le \frac{|x|^5}{5!} \le 10^{-5} \quad \text{or} \quad |x| \le \frac{\sqrt[5]{120}}{10}.$$

Hence we can take $d = \frac{\sqrt[5]{120}}{10}$.

7. (15 points) Plot the points whose polar coordinates are given below

$$(1, 3\pi/2), \quad (-2, 5\pi/4), \quad (-1, -3\pi/2).$$

Solution: Using the coordinates conversion $x = r\cos\theta$ and $y = r\sin\theta$, we find that in Cartesian coordinates the first point is $(-1, 0)$, the second is $(\sqrt{2}, \sqrt{2})$, and the third one is $(0, -1)$.

8. (10 points) Convert the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ from Cartesian to polar coordinates. Give at least two different representations in polar coordinates.

Solution:

Using the coordinates conversion

$$x^2 + y^2 = r^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

we find

$$r = \sqrt{1/4 + 3/4} = 1 \quad \text{and} \quad \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

Hence the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ in Cartesian coordinates corresponds to a point $1, \pi/3)$ in polar coordinates. Same point in polar coordinates we can write as $(1, 7\pi/3)$ or $(-1, 4\pi/3)$, and etc.

9. (20 points) Let the curve C be defined by the parametric equations $x = t^2 - t$ and $y = t^3 + 1$. Find the equation of the tangent line when $t = 2$. Find the points on the curve where the tangent line is horizontal or vertical.

Solution:

When $t = 2$, $x = 2^2 - 2 = 2$ and $y = 2^3 + 1 = 9$. Thus for the equation of the tangent line we only need the slope. Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 1}{3t^2},$$

hence the slope at $t = 2$ is $\frac{2\cdot 2 - 1}{3\cdot 2^2} = \frac{3}{12} = \frac{1}{4}$ and the equation of the tangent line is

$$y - 9 = \frac{1}{4}(x - 2).$$

The tangent line is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \ne 0$ and horizontal if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \ne 0$. Thus,

$$\frac{dx}{dt} = 0 \quad \Longrightarrow \quad 3t^2 = 0 \quad \Longrightarrow \quad t = 0$$

and

$$\frac{dy}{dt} = 0 \quad \Longrightarrow \quad 2t - 1 = 0 \quad \Longrightarrow \quad t = 1/2.$$

Thus, the tangent line is horizontal at $(-1/4, 9/8)$ and vertical at $(0, 1)$.

10. (10 points) Using Taylor formula, compute the following limits.

(a)

$$\lim_{x\to 0}\frac{e^{2x}-1-2x-2x^2}{x^3}$$

(b)

$$\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}-\frac{x^4}{24}}{x^6}$$

Solution:

(a) Using MacLaurin series for e^{2x}

$$e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}=1+2x+2x^2+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\cdots$$

we find

$$\lim_{x\to 0}\frac{e^{2x}-1-2x-2x^2}{x^3}=\frac{8}{3!}+\frac{16x}{4!}+\cdots=\frac{4}{3}.$$

(b) Using MacLaurin series for $\cos x$

$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

we find

$$\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}-\frac{x^4}{24}}{x^6}=-\frac{1}{6!}.$$