MATH 1132 – Calculus II

Practice Final Exam Solutions

No calculators. Show your work. Clearly mark each answer.

1. Find the limits of the following sequences $\{a_n\}$.

   (a)
$$a_n = ne^{-n}$$

   (b)
$$a_n = \frac{\cos n}{\sqrt{n}}$$

   (c)
$$a_n = \left(1 + \frac{1}{n}\right)^{2n}$$

   Solution:

   (a) Since by the L'Hospital rule
$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$
   it follows that
$$\lim_{n\to\infty} ne^{-n} = 0.$$

   (b) Since $-1 \leq \cos x \leq 1$ for any x, we have
$$-\frac{1}{\sqrt{n}} \leq \frac{\cos n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$
   Because $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ by the Sandwich theorem it follows that
$$\lim_{n\to\infty} \frac{\cos n}{\sqrt{n}} = 0.$$

   (c) Using that
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$
   we find that
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^n\right)^2 = \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right)^2 = e^2.$$

2. Which of the following series converge or diverge? Give reasons to your answers.

   (a)
$$\sum_{n=2}^{\infty} \frac{n^2+1}{(n^2-1)^2}$$

   (b)
$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$$

(c)

$$\sum_{n=0}^{\infty} \frac{n}{(n+1)^4}$$

Solution:

(a) Since terms of the series are positive we can use the Limit Comparison Test. The leading term in the numerator is n^2 and n^4 in the denominator we compare the original series to $\sum \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{(n^2-1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(n^2+1)}{(n^2-1)^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{(1-\frac{1}{n^2})^2} = 1 > 0$$

and we know that $\sum \frac{1}{n^2}$ converges as a p-series with $p = 2$, it follows that $\sum_{n=2}^{\infty} \frac{n^2+1}{(n^2-1)^2}$ converges as well.

(b) This is an alternating series. To show convergence we need to show that $\frac{1}{n \ln n}$ monotonically going to 0. Obviously

$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$$

and $x \ln x$ is monotonically increasing function (since both x and $\ln x$ are). Thus by the Alternating Series Theorem $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges.

(c) By the direct comparison test

$$\frac{n}{(n+1)^4} \leq \frac{n}{n^4} = \frac{1}{n^3}$$

and we know that $\sum \frac{1}{n^3}$ converges as a p-series with $p = 3$, it follows that $\sum_{n=0}^{\infty} \frac{1}{(n+1)^4}$ converges as well.

3. Find the radii and the intervals of convergence of the following power series.

(a)

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n^2}$$

(b)

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n}$$

(c)

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$$

Solutions:

(a) By the ratio test we need to have

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}x^{n+1}}{(n+1)^2}}{\frac{3^n x^n}{n^2}} \right| = \lim_{n\to\infty} 3|x| \frac{n+1}{n} 3|x| \lim_{n\to\infty} \frac{n+1}{n} = 3|x| < 1.$$

Thus $|x| < 1/3$ and the radius of convergence $R = \frac{1}{3}$. To figure out the interval of convergence we need to check the convergence for $x = \pm\frac{1}{3}$.

For $x = \frac{1}{3}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

by the p-series test. Similarly for $x = -\frac{1}{3}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} < \infty$$

by the Absolute convergence theorem or the Alternating series theorem. Thus, the interval of convergence is $[-\frac{1}{3}, \frac{1}{3}]$.

(b) By the n-Root Test we need to have

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x-1|^n}{2^n n}} = \frac{|x-1|}{2} \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{|x-1|}{2} < 1,$$

or $|x-1| < 2$. Thus the radius of convergence is 2. To find out the interval of convergence we need to check the convergence of the series for $|x-1| = 2$ or $x = 3$ and $x = -1$.
For $x = 3$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

the Harmonic series and hence diverges. For $x = -1$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} < \infty$$

by the Alternating Series Theorem. Thus, the interval of convergence is $[-1, 3)$.

(c) By the n-Root Test we need to have

$$\lim_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{4^n}} = \frac{x^2}{4} < 1,$$

or $x^2 < 4$ or $|x| < 2$. Thus the radius of convergence is 2. To find out the interval of convergence we need to check the convergence of the series for $|x| = 2$ or $x = 2$ and $x = -2$. For $x = 2$ we have

$$\sum_{n=0}^{\infty} 1$$

obviously diverges. Similarly for $x = -2$ we have

$$\sum_{n=0}^{\infty} 1 = \infty.$$

Thus the interval of convergence is $(-2, 2)$.

4. Compute the following limits.

(a)

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2}$$

(b)

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

Solutions:

(a) This limit is in the indeterminate $\frac{0}{0}$ form. Applying the L'Hospital rule we have

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=\lim_{x\to 0}\frac{2e^{2x}-2}{2x},$$

which is again in the indeterminate $\frac{0}{0}$ form. Applying the L'Hospital rule once again we finally compute

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=\lim_{x\to 0}\frac{2e^{2x}-2}{2x}=\lim_{x\to 0}\frac{4e^{2x}}{2}=2.$$

(b) As above we can use L'Hospital rule. However we know that the Maclaurin series for $\sin x$ is

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots,$$

Hence,

$$\lim_{x\to 0}\frac{\sin x - x+\frac{x^3}{6}}{x^5}=\lim_{x\to 0}\frac{\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots}{x^5}=\lim_{x\to 0}\frac{1}{5!}-\frac{x^2}{7!}+\cdots=\frac{1}{120}.$$

5. Find the Maclaurin series of the following function

$$(x-1)e^{3x}.$$

Solutions: Using that the Maclaurin series for e^x

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

we find that

$$e^{3x}=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}$$

and

$$\begin{aligned}(x-1)e^{3x}&=xe^{3x}-e^{3x}=\sum_{n=0}^{\infty}\frac{3^n x^{n+1}}{n!}-\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}\\&=\sum_{n=1}^{\infty}\frac{3^{n-1}x^n}{(n-1)!}-\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}\\&=-1+\sum_{n=1}^{\infty}\left(\frac{3^{n-1}}{(n-1)!}-\frac{3^n}{n!}\right)x^n.\end{aligned}$$

6. Find the quadratic ($n=2$) Taylor polynomial at $a=1$ of the following function

$$x^{5/2}.$$

Solution: Looking at the Taylor series formula

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n,$$

we can see that to derive the quadratic Taylor polynomial at $a=2$ we need to compute $f(1)$, $f'(1)$, and $f''(1)$. Thus,

$$\begin{aligned}f(1)&=1\\ f'(x)=\frac{5}{2}x^{3/2},\ \Rightarrow\ f'(1)&=\frac{5}{2}\\ f''(x)=\frac{15}{4}x^{1/2},\ \Rightarrow\ f''(1)&=\frac{15}{4},\end{aligned}$$

and the quadratic polynomial at $a=1$ is

$$T_2(x)=1+\frac{5}{2}(x-1)+\frac{15}{4}\frac{(x-1)^2}{2}.$$

7. What is the largest d can be such that the approximation

$$\cos x \approx 1 - x^2/2$$

is accurate to 4 decimal places for $|x| \le d$?

Solution: We need find largest d such that the remainder $R_2(x) = \cos x - (1 - x^2/2)$ is bounded by 10^{-4} for $|x| \le d$. Since the Maclaurin series for cosine

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is alternating, we have

$$|R_2(x)| \le \frac{|x|^4}{4!}.$$

Thus we want,

$$|R_2(x)| \le \frac{|x|^4}{4!} \le 10^{-4} \quad \text{or} \quad |x| \le \frac{\sqrt[4]{24}}{10}.$$

Hence we can take $d = \frac{\sqrt[4]{24}}{10}$.

8. Plot the points whose polar coordinates are given below

$$(2, \pi/2), \quad (-1, 3\pi/4), \quad (-1, -\pi/2).$$

Solution: For example using the coordinates conversion $x = r\cos\theta$ and $y = r\sin\theta$, we find that in Cartesian coordinates the first point is $(0, 2)$, the second is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and the third one is $(0, 1)$.

9. Convert the point $(1, 1)$ from Cartesian to polar coordinates. Give at least two different representations in polar coordinates.

Solution:

Using the coordinates conversion

$$x^2 + y^2 = r^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

we find

$$r = \sqrt{1+1} = \sqrt{2} \quad \text{and} \quad \theta = \tan^{-1}(1) = \frac{\pi}{4}.$$

Hence the point $(1, 1)$ in Cartesian coordinates corresponds to a point $(\sqrt{2}, \pi/4)$ in polar coordinates. Same point in polar coordinates we can write as $(\sqrt{2}, 9\pi/4)$ or $(-\sqrt{2}, 5\pi/4)$, and etc.

10. Let the curve C be defined by the parametric equations $x = t^3 - t$ and $y = t^2 - 1$. Find the equation of the tangent line when $t = 2$. Find the points on the curve where the tangent line is horizontal or vertical.

Solution:

When $t = 2$, $x = 2^3 - 2 = 6$ and $y = 2^2 - 1 = 3$. Thus for the equation of the tangent line we only need the slope. Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 - 1},$$

hence the slope at $t = 2$ is $\frac{2 \cdot 2}{3 \cdot 2^2 - 1} = \frac{4}{11}$ and the equation of the tangent line is

$$y - 3 = \frac{4}{11}(x - 6).$$

The tangent line is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ and horizontal if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. Thus,

$$\frac{dx}{dt} = 0 \quad \Longrightarrow \quad 3t^2 - 1 = 0 \quad \Longrightarrow \quad t = \pm\frac{1}{\sqrt{3}}.$$

and

$$\frac{dy}{dt} = 0 \quad \Longrightarrow \quad 2t = 0 \quad \Longrightarrow \quad t = 0.$$

Thus, the tangent line is horizontal at $(0, -1)$ and vertical at $(-\frac{2}{3\sqrt{3}}, -\frac{2}{3})$ and $(\frac{2}{3\sqrt{3}}, -\frac{2}{3})$.

11. Find the area of the region enclosed by one loop of the curve $r = \sin(4\theta)$.

Solution: By the formula

$$A = \int_a^b \frac{1}{2} r^2(\theta) d\theta,$$

with $a = 0$ and $b = \frac{\pi}{4}$ we find

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin^2(4\theta) d\theta = \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 - \cos(8\theta))\, d\theta \\ &= \frac{1}{4} \left(\theta - \frac{1}{8} \sin(8\theta) \right) \Big|_0^{\pi/4} = \frac{\pi}{16}. \end{aligned}$$