Practice Exam 3. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. Find the radii and the intervals of convergence of the following power series.

(a)

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n}$$

Solution. Using the n-Root Test we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-1|}{2\sqrt[n]{n}} = \frac{|x-1|}{2},$$

since $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Thus for the series to converge we need

$$\frac{|x-1|}{2} < 1 \quad \Longrightarrow \quad |x-1| < 2 \quad \Longrightarrow \quad -1 < x < 3.$$

Thus the radius of convergence is 2. To find the interval of convergence we need to check the end points $x = -1$ and $x = 3$. When $x = 3$ we have

$$\sum_{n=1}^{\infty} \frac{(3-1)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

a Harmonic series which diverges. When $x = -1$ we have

$$\sum_{n=1}^{\infty} \frac{(-1-1)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

an alternating series which converges by the Alternating Series Theorem. Thus the interval of convergence is $[-1, 3)$.

(b)

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$$

Solution. Using the n-Root Test we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x|^2}{4} = \frac{|x|^2}{4}.$$

Thus for the series to converge we need

$$\frac{|x|^2}{4} < 1 \quad \Longrightarrow \quad |x|^2 < 4 \quad \Longrightarrow \quad |x| < 2 \quad \Longrightarrow \quad -2 < x < 2.$$

Thus the radius of convergence is 2. To find the interval of convergence we need to check the end points $x = -2$ and $x = 2$. When $x = 2$ we have

$$\sum_{n=1}^{\infty} \frac{(2)^{2n}}{4^n} = \sum_{n=1}^{\infty} 1$$

which diverges. When $x = -2$ we have again

$$\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{4^n} = \sum_{n=1}^{\infty} 1$$

which diverges again. Thus the interval of convergence is $(-2, 2)$.

(c)

$$\sum_{n=0}^{\infty} \frac{n^2(x+2)^n}{2^n}$$

Solution. Using the n-Root Test we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}|x+2|}{2} = \frac{|x+2|}{2},$$

since $\lim_{n\to\infty} \sqrt[n]{n^2} = 1$. Thus for the series to converge we need

$$\frac{|x+2|}{2} < 1 \quad \Longrightarrow \quad |x+2| < 2 \quad \Longrightarrow \quad -4 < x < 0.$$

Thus the radius of convergence is 2. To find the interval of convergence we need to check the end points $x = -4$ and $x = 0$. When $x = 0$ we have

$$\sum_{n=1}^{\infty} \frac{n^2 2^n}{2^n} = \sum_{n=1}^{\infty} n^2$$

which diverges. When $x = -4$ we have

$$\sum_{n=1}^{\infty} \frac{n^2(-1-1)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n^2$$

an alternating series which diverges since n-term does not go to zero. Thus the interval of convergence is $(-4, 0)$.

2. Using Maclaurin series, compute the following limits.

(a)

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2}$$

Solution. Since the Maclaurin series for e^{2x} is

$$1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

we have

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2} = \lim_{x\to 0} \frac{\frac{(2x)^2}{2} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots}{x^2} = \lim_{x\to 0} \left(\frac{2^2}{2} + \frac{(2)^3 x}{3!} + \frac{(2)^4 x^2}{4!} + \cdots \right) = 2.$$

(b)

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

Solution. Since the Maclaurin series for $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

we have

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x^5} = \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots \right) = \frac{1}{5!} = \frac{1}{120}.$$

3. Find the quadratic ($n = 2$) Taylor polynomial at $a = 1$ of the following function

$$x^{3/2}.$$

Solution. To use the formula

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2,$$

we need to compute $f(1)$, $f'(1)$ and $f''(1)$.
Since $f(x) = x^{3/2}$, $f'(x) = \frac{3}{2}x^{1/2}$ and $f''(x) = \frac{3}{4}x^{-1/2}$, we have $f(1) = 1$, $f'(1) = \frac{3}{2}$ and $f''(1) = \frac{3}{4}$. Thus

$$T_2(x) = 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2.$$

4. What is the largest d can be such that the approximation

$$\cos x \approx 1 - x^2/2$$

is accurate to 4 decimal places for $|x| \leq d$?

Solution. Since the Maclaurin series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!},$$

it is alternating series, so we can estimate the remainder $R_2(x) = \cos x - T_2(x) = \cos x - (1 - x^2/2)$ as

$$|R_2(x)| \leq \frac{|x|^4}{4!}.$$

Thus we need to find largest $d > 0$ such that $\frac{d^4}{4!} \leq 10^{-4}$. Solving for d we find

$$d^4 \leq \frac{4!}{10^4} \quad \Longrightarrow \quad d \leq \frac{\sqrt[4]{24}}{10}.$$

Hence the largest $d = \frac{\sqrt[4]{24}}{10}$.

5. Find the Maclaurin series of the following functions and find the radii of convergence.

   (a)

$$\sin(x^2/4)$$

Solution.
Since the Maclaurin series for $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

and has infinite radius of convergence, we have

$$\sin(x^2/4) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2/4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2^{4n+2}(2n+1)!},$$

and also has infinite radius of convergence.

(b)

$$\ln(1-4x)$$

Solution.
Since

$$\frac{d}{dx}\ln(1-4x) = \frac{-4}{1-4x}$$

we know the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

we have

$$\frac{-4}{1-4x} = -4\sum_{n=0}^{\infty}(4x)^n, \quad |x| < 1/4, .$$

Integrating we obtain

$$\ln(1-4x) = \int \frac{-4dx}{1-4x} = C - 4\sum_{n=0}^{\infty}\frac{4^n x^{n+1}}{n+1}.$$

Taking $x = 0$ in the above expression and noting that since $\ln 1 = 0$, we have $C = 0$ and as a result

$$\ln(1-4x) = -4\sum_{n=0}^{\infty}\frac{4^n x^{n+1}}{n+1}, \quad R = \frac{1}{4}.$$

6. Find the Taylor series of the following functions at $a = 1$

(a)

$$\frac{1}{x}$$

Solution.
To use the Taylor formula

$$f(x) \sim \sum_{n=0}^{\infty}\frac{f^{(n)}(1)(x-1)^n}{n!},$$

with $f(x) = 1/x$, we need to compute $f^{(n)}(1)$ for all n. Since

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}},$$

we have $f^{(n)}(1) = (-1)^n n!$, and as a result the Taylor formula is

$$\frac{1}{x} \sim \sum_{n=0}^{\infty}(-1)^n(x-1)^n.$$

(b)

$$xe^{2x}$$

Solution.
Since the Taylor series for e^{2x} at $a = 1$ is

$$e^x = \sum_{n=0}^{\infty}\frac{f^n(1)(x-1)^n}{n!},$$

and

$$\frac{d^n}{dx^n}\left(e^{2x}\right) = 2^n e^{2x},$$

we obtain

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^2 (x-1)^n}{n!}.$$

Since $x = 1 + (x-1)$, we obtain

$$\begin{aligned}
xe^{2x} &= (1 + (x-1)) \sum_{n=0}^{\infty} \frac{2^n e^2 (x-1)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{2^n e^2 (x-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{2^n e^2 (x-1)^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{2^n e^2 (x-1)^n}{n!} + \sum_{n=1}^{\infty} \frac{2^{n-1} e^2 (x-1)^n}{(n-1)!} \\
&= e^2 + e^2 \sum_{n=1}^{\infty} \frac{2^n + n2^{n-1}}{n!} (x-1)^n.
\end{aligned}$$