MATH 1132 – Calculus II March 10, 2020

Practice Exam 2

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Find the limits of the following sequences $\{a_n\}$.

   (a)
$$a_n = \frac{n^2 + 5n - 1}{3n^3 + 5}$$

   (b)
$$a_n = \frac{\sin n}{\sqrt[3]{n}}$$

   (c)
$$a_n = \frac{6n^3 - n^2 + 1}{2n^3 + 5n}$$

   (d)
$$a_n = \frac{\ln n}{n}$$

Solution:

(a) Dividing numerator and denominator by n^3 we obtain

$$a_n = \frac{n^2 + 5n - 1}{3n^3 + 5} = \frac{\frac{1}{n} + \frac{5}{n^2} - \frac{1}{n^3}}{3 + \frac{5}{n^3}}.$$

Hence

$$\lim_{n\to\infty} a_n = \frac{0 - 0 + 0}{3 + 0} = 0.$$

(b) Since $-1 \le \sin x \le 1$ for any x, we have

$$-\frac{1}{\sqrt[3]{n}} \le \frac{\sin n}{\sqrt[3]{n}} \le \frac{1}{\sqrt[3]{n}}.$$

Because $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$ by the Squeeze (Sandwich) theorem it follows that

$$\lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0.$$

(c) We can rewrite a_n as

$$a_n = \left(1 + \frac{3}{n}\right)^n = \left(\left(1 + \frac{3}{n}\right)^{\frac{n}{3}}\right)^3.$$

Hence putting $m = n/3$

$$\lim_{n\to\infty} a_n = \left(\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right)^3 = e^3.$$

(d) Since by the L'Hospital's Rule

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

we can conclude that

$$a_n = \frac{\ln n}{n} = 0.$$

2. (20 points) Which of the following series converge or diverge? Give reasons to your answers.

(a)

$$\sum_{n=2}^{\infty} \frac{n^3+1}{(n^3-1)^2}$$

(b)

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$$

(c)

$$\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{n+1}$$

(d)

$$\sum_{n=1}^{\infty} \frac{5^n}{n!}$$

Solution:

(a) Since terms of the series are positive we can use the Limit Comparison Test. The leading term in the numerator is n^3 and n^6 in the denominator, so we compare the original series to $\sum \frac{1}{n^3}$. Since

$$\lim_{n\to\infty} \frac{\frac{n^3+1}{(n^3-1)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3(n^3+1)}{(n^3-1)^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n^3}}{(1-\frac{1}{n^3})^2} = 1 > 0$$

and we know that $\sum \frac{1}{n^3}$ converges as a p-series with $p = 3$, it follows that $\sum_{n=2}^{\infty} \frac{n^3+1}{(n^3-1)^2}$ converges as well.

(b) This is an alternating series. To show convergence we need to show that $\frac{1}{\ln n}$ monotonically going to 0. Obviously

$$\lim_{n\to\infty} \frac{1}{\ln n} = 0$$

and $\ln x$ is monotonically increasing function . Thus by the Alternating Series Theorem $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$ converges.

(c) Since

$$\lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \infty,$$

the necessary condition for the convergence of the series is not satisfied and hence the series diverges.

(d) Since terms of the series are positive we can use the Ratio Comparison Test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} = \lim_{n\to\infty} \frac{5^n 5}{5^n} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1.$$

Hence the series converges.

3. (20 points) Using partial fractions show that the following improper integral converges.

$$\int_1^\infty \frac{dx}{x^2(x+1)}$$

What can you say about convergence of the following series?

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

Solution:

Since $\frac{1}{x^2(x+1)} = -\frac{x-1}{x^2} + \frac{1}{x+1}$, we have

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2(x+1)} &= \lim_{M\to\infty} \int_1^M \left(-\frac{x-1}{x^2} + \frac{1}{x+1}\right) dx \\
&= \lim_{M\to\infty} \int_1^M \left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right) dx \\
&= \lim_{M\to\infty} \left[-\frac{1}{x} - \ln x + \ln(x+1)\right] |_1^M \\
&= \lim_{M\to\infty} \left(\frac{1}{M} + \ln\left(\frac{M+1}{M}\right)\right) + 1 - \ln 2 \\
&= 1 - \ln 2.
\end{aligned}$$

Since the improper integral converges, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

consisting of positive terms by the integral test converges as well.

4. (20 points) Express the following number as a ratio of two integers.

$$0.113113113113\ldots$$

Solution: We can rewrite the number as

$$0.113113113113\cdots = \frac{113}{10^3} + \frac{113}{10^6} + \frac{113}{10^9} + \cdots = 113 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{1000}\right)^n.$$

Using the properties of geometric series $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ with $r = \frac{1}{1000}$, we obtain,

$$0.113113113113\cdots = 113 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{1000}\right)^n = 113 \cdot \frac{\frac{1}{1000}}{1 - \frac{1}{1000}} = \frac{113}{999}.$$

5. (20 points) Find the value of the following series

$$\sum_{n=2}^{\infty} \left(\frac{2}{2^n} - \frac{3}{3^n}\right).$$

Solution: We can rewrite the series as

$$\sum_{n=2}^{\infty} \frac{2}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

and

$$\sum_{n=2}^{\infty} \frac{3}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n .$$

Using the properties of geometric series $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ with $r = \frac{1}{2}$ and $r = \frac{1}{3}$, we obtain,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$$

hence

$$\sum_{n=2}^{\infty} \left(\frac{2}{2^n} - \frac{3}{3^n}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$