

Practice Exam 1. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Find the area of the region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = x + 2.$$

Sketch the area.

Solution: From the equation $x^2 = x + 2$ we find that the parabola $y = x^2$ and the straight line $y = x + 2$ intersect at $x = -1$ and $x = 2$. The sketch of the region is below. Hence the area of region is

$$A = \int_{-1}^2 (x + 2 - x^2) \, dx = \left. \frac{(x + 2)^2}{2} - \frac{x^3}{3} \right|_{-1}^2 = 8 - \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{9}{2} = 4.5.$$

2. (20 points) Find the volume of the solid obtained by rotating the region bounded by curves

$$y = e^x, \quad x = 0, \quad x = 1, \quad y = 0,$$

about $y = -1$ line. Sketch the region.

Solution: Slicing the solid along y -axis (see the sketch below), we have a washer with the small radius $r = 1$ and the larger radius $R = e^x + 1$ (which is a distance from $y = e^x$ to $y = -1$). Hence

$$\begin{aligned} \text{Volume} &= \pi \int_0^1 (R^2(x) - r^2(x)) \, dx = \pi \int_0^1 ((e^x + 1)^2 - 1) \, dx = \pi \int_0^1 (e^{2x} + 2e^x) \, dx \\ &= \pi \left(\frac{e^{2x}}{2} + 2e^x \right) \Big|_0^1 = \pi \left(\frac{e^2}{2} + 2e - \frac{1}{2} - 2 \right) = \pi \left(\frac{e^2}{2} + 2e - \frac{5}{2} \right). \end{aligned}$$

3. (20 points) Using integration by part, evaluate the following integrals:

(a)

$$\int \ln^2 x \, dx$$

Solution: Taking $u = \ln^2 x$, $dv = dx$ and as a result $du = \frac{2 \ln x}{x}$, $v = x$ we obtain

$$\int \ln^2 x \, dx = x \ln^2 x - 2 \int \ln x \, dx.$$

To find $\int \ln x \, dx$, we use integration by parts again with $u = \ln x$ and $dv = dx$

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C.$$

Thus,

$$\int \ln^2 x \, dx = x \ln^2 x - 2 \int \ln x \, dx = x \ln^2 x - 2x \ln x + 2x + C.$$

(b)

$$\int_0^1 x^2 e^{-x} dx$$

Solution: Taking $u = x^2$, $dv = e^{-x} dx$ and as a result $du = 2x$, $v = -e^{-x}$, we obtain

$$\int_0^1 x^2 e^{-x} dx = -x^2 e^{-x} \Big|_0^1 + 2 \int_0^1 x e^{-x} dx = -e^{-1} + 2 \int_0^1 x e^{-x} dx.$$

To compute $2 \int_0^1 x e^{-x} dx$, we use integration by parts again with $u = x$, $dv = e^{-x} dx$, hence $du = dx$ and $v = -e^{-x}$

$$2 \int_0^1 x e^{-x} dx = -2x e^{-x} \Big|_0^1 + 2 \int_0^1 e^{-x} dx = -2e^{-1} - 2e^{-x} \Big|_0^1 = -2e^{-1} - 2e^{-1} + 2 = 2 - 4e^{-1}.$$

Combining the above calculations, we obtain

$$\int_0^1 x^2 e^{-x} dx = -e^{-1} + 2 - 4e^{-1} = 2 - 5e^{-1}.$$

4. (10 points) Using a trigonometric substitution, evaluate the following integral:

$$\int \frac{1}{(16 + x^2)^{3/2}} dx$$

Solution: Using a change of variables $x = 4 \tan \theta$, $dx = 4 \sec^2 \theta d\theta$, we obtain

$$\int \frac{1}{(16 + x^2)^{3/2}} dx = \int \frac{4 \sec^2 \theta}{4^3 \sec^3 \theta} d\theta = \frac{1}{16} \int \cos \theta d\theta = \frac{1}{16} \sin \theta + C.$$

Now, from the right triangle we find out that if $\tan \theta = \frac{x}{4}$, then $\sin \theta = \frac{x}{\sqrt{x^2 + 16}}$ and as a result

$$\int \frac{1}{(16 + x^2)^{3/2}} dx = \frac{1}{16} \frac{x}{\sqrt{x^2 + 16}} + C.$$

5. (10 points) Evaluate the following integral:

$$\int \sin^2(x) \sin(2x) dx$$

Solution: Using double angle formula

$$\sin(2x) = 2 \sin(x) \cos(x),$$

we have

$$\int \sin^2(x) \sin(2x) dx = 2 \int \sin^3(x) \cos(x) dx.$$

Using the change of variable $z = \sin x$ and $dz = \cos x dx$ we compute

$$2 \int \sin^3(x) \cos(x) dx = 2 \int z^3 dz = \frac{2}{4} z^4 + C = \frac{1}{2} \sin^4(x) + C.$$

Combining, we have

$$\int \sin^2(x) \sin(2x) dx = \frac{1}{2} \sin^4(x) + C.$$

6. (20 points) Using partial fractions, evaluate the following integrals:

(a)

$$\int \frac{2}{x(x-2)(x+1)} dx$$

Solution: Since all the factors are linear we have

$$\frac{2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Thus,

$$2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

for some numbers A , B , and C that we need to find.

Setting $x = 0$ in the above expression, we have $2 = -2A$. Thus, $A = -1$.

Setting $x = 2$ in the above expression, we have $2 = 6B$. Thus, $B = \frac{1}{3}$.

Finally, setting $x = -1$ in the above expression, we have $2 = 3C$. Thus, $C = \frac{2}{3}$.

As a result

$$\int \frac{2}{x(x-2)(x+1)} dx = \int \left(-\frac{1}{x} + \frac{1}{3} \frac{1}{x-2} + \frac{2}{3} \frac{1}{x+1} \right) dx = -\ln|x| + \frac{1}{3} \ln|x-2| + \frac{2}{3} \ln|x+1| + C.$$

(b)

$$\int \frac{x}{(x+1)(x^2+4)} dx$$

Solution:

Since $x+1$ is linear and x^2+4 is irreducible quadratic, we have

$$\frac{x}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4) + (Bx+C)(x+1)}{(x+1)(x^2+4)},$$

Thus,

$$x = A(x^2+4) + (x+1)(Bx+C)$$

for some numbers A , B , and C that we need to find.

Setting $x = -1$ in the above expression, we have $-1 = 5A$. Thus, $A = -\frac{1}{5}$. Inserting it into the above equation we find

$$(x+1)(Bx+C) = x + \frac{1}{5}(x^2+4) = \frac{x^2+5x+4}{5} = \frac{(x+4)(x+1)}{5}$$

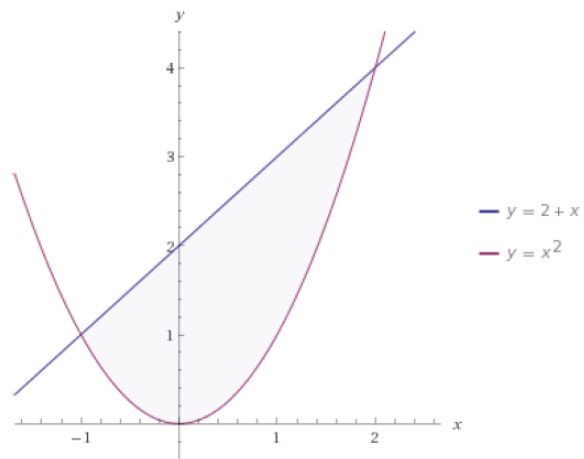
or

$$Bx+C = \frac{x+4}{5},$$

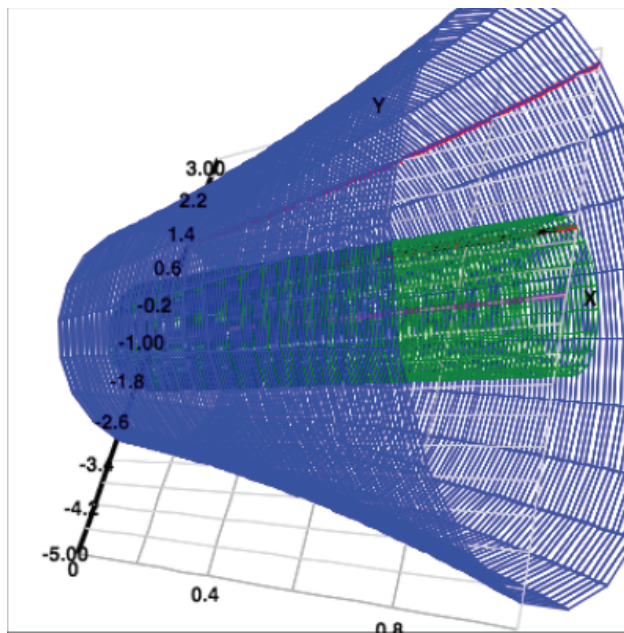
and as a result $B = \frac{1}{5}$ and $C = \frac{4}{5}$. Thus

$$\begin{aligned} \int \frac{x}{(x+1)(x^2+4)} dx &= \int \left(-\frac{1}{5} \frac{1}{x+1} + \frac{1}{5} \frac{x+4}{x^2+4} \right) dx \\ &= -\frac{1}{5} \int \frac{dx}{x+1} + \frac{1}{5} \int \frac{x}{x^2+4} dx + \frac{4}{5} \int \frac{dx}{x^2+4} \\ &= -\frac{1}{5} \ln|x+1| + \frac{1}{10} \ln(x^2+4) + \frac{2}{5} \tan^{-1} \left(\frac{x}{2} \right) + C. \end{aligned}$$

Sketch of the area between curves $y = x + 2$ and $y = x^2$



Sketch of the volume of revolution of $y = e^x$ and $y = 0$ about the axis $y = -1$



7. (10 points) Evaluate the following improper integral

$$\int_1^{\infty} \frac{dx}{(x+1)^2}.$$

Solution:

By the definition

$$\int_1^{\infty} \frac{dx}{(x+1)^2} = \lim_{M \rightarrow \infty} \int_1^M \frac{dx}{(x+1)^2}.$$

Using the Fundamental Theorem of Calculus

$$\int_1^M \frac{dx}{(x+1)^2} = -\frac{1}{x+1} \Big|_1^M = -\frac{1}{M+1} + \frac{1}{2}.$$

Since

$$\lim_{M \rightarrow \infty} \left(-\frac{1}{M+1} + \frac{1}{2} \right) = \frac{1}{2},$$

we obtain

$$\int_1^{\infty} \frac{dx}{(x+1)^2} = \frac{1}{2}.$$