MATH 1132 – Calculus II February 13, 2020

Practice Exam 1. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. (20 points) Find the area of the region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = x + 2.$$

Sketch the area.

Solution: From the equation $x^2 = x + 2$ we find that the parabola $y = x^2$ and the straight line $y = x+2$ intersect at $x = -1$ and $x = 2$. The sketch of the region is below. Hence the area of region is

$$A = \int_{-1}^{2} (x + 2 - x^2)\ dx = \frac{(x+2)^2}{2} - \frac{x^3}{3}\bigg|_{-1}^{2} = 8 - \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{9}{2} = 4.5.$$

2. (20 points) Find the volume of the solid obtained by rotating the region bounded by curves

$$y = e^x,\ x = 0,\ x = 1,\ y = 0,$$

about $y = -1$ line. Sketch the region.

Solution: Slicing the solid along y-axis (see the sketch below), we have a washer with the small radius $r = 1$ and the larger radius $R = e^x + 1$ (which is a distance from $y = e^x$ to $y = -1$). Hence

$$Volume = \pi \int_0^1 (R^2(x) - r^2(x))\ dx = \pi \int_0^1 ((e^x+1)^2 - 1)\ dx = \pi \int_0^1 (e^{2x} + 2e^x)\ dx$$
$$= \pi \left(\frac{e^{2x}}{2} + 2e^x\right)\bigg|_0^1 = \pi \left(\frac{e^2}{2} + 2e - \frac{1}{2} - 2\right) = \pi \left(\frac{e^2}{2} + 2e - \frac{5}{2}\right).$$

3. (20 points) Using integration by part, evaluate the following integrals:

(a)

$$\int \ln^2 x\ dx$$

Solution: Taking $u = \ln^2 x$, $dv = dx$ and as a result $du = \frac{2\ln x}{x}$, $v = x$ we obtain

$$\int \ln^2 x\ dx = x \ln^2 x - 2 \int \ln x\ dx.$$

To find $\int \ln x\ dx$, we use integration by parts again with $u = \ln x$ and $dv = dx$

$$\int \ln x\ dx = x \ln x - \int dx = x \ln x - x + C.$$

Thus,

$$\int \ln^2 x\ dx = x \ln^2 x - 2 \int \ln x\ dx = x \ln^2 x - 2x \ln x + 2x + C.$$

(b)

$$\int_0^1 x^2 e^{-x}\ dx$$

Solution: Taking $u = x^2$, $dv = e^{-x}dx$ and as a result $du = 2x$, $v = -e^{-x}$, we obtain

$$\int_0^1 x^2 e^{-x}\ dx = -x^2 e^{-x}|_0^1 + 2\int_0^1 xe^{-x}\ dx = -e^{-1} + 2\int_0^1 xe^{-x}\ dx.$$

To compute $2\int_0^1 xe^{-x}\ dx$, we use intergration by parts again with $u = x$, $dv = e^{-x}dx$, hence $du = dx$ and $v = -e^{-x}$

$$2\int_0^1 xe^{-x}\ dx = -2xe^{-x}|_0^1 + 2\int_0^1 e^{-x}\ dx = -2e^{-1} - 2e^{-x}|_0^1 = -2e^{-1} - 2e^{-1} + 2 = 2 - 4e^{-1}.$$

Combining the above calculations, we obtain

$$\int_0^1 x^2 e^{-x}\ dx = -e^{-1} + 2 - 4e^{-1} = 2 - 5e^{-1}.$$

4. (10 points) Using a trigonometric substitution, evaluate the following integral:

$$\int \frac{1}{(16+x^2)^{3/2}}\ dx$$

Solution: Using a change of variables $x = 4\tan\theta$, $dx = 4\sec^2\theta d\theta$, we obtain

$$\int \frac{1}{(16+x^2)^{3/2}}\ dx = \int \frac{4\sec^2\theta}{4^3\sec^3\theta}\ d\theta = \frac{1}{16}\int \cos\theta\ d\theta = \frac{1}{16}\sin\theta + C.$$

Now, from the right triangle we find out that if $\tan\theta = \frac{x}{4}$, then $\sin\theta = \frac{x}{\sqrt{x^2+16}}$ and as a result

$$\int \frac{1}{(16+x^2)^{3/2}}\ dx = \frac{1}{16}\frac{x}{\sqrt{x^2+16}} + C.$$

5. (10 points) Evaluate the following integral:

$$\int \sin^2(x)\sin(2x)\ dx$$

Solution: Using double angle formula

$$\sin(2x) = 2\sin(x)\cos(x),$$

we have

$$\int \sin^2(x)\sin(2x)\ dx = 2\int \sin^3(x)\cos(x)\ dx.$$

Using the change of variable $z = \sin x$ and $dz = \cos x dx$ we compute

$$2\int \sin^3(x)\cos(x)\ dx = 2\int z^3\ dz = \frac{2}{4}z^4 + C = \frac{1}{2}\sin^4(x) + C.$$

Combining, we have

$$\int \sin^2(x)\sin(2x)\ dx = \frac{1}{2}\sin^4(x) + C.$$

6. (20 points) Using partial fractions, evaluate the following integrals:

(a)

$$\int \frac{2}{x(x-2)(x+1)}\,dx$$

Solution: Since all the factors are linear we have

$$\frac{2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Thus,

$$2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

for some numbers A, B, and C that we need to find.
Setting $x = 0$ in the above expression, we have $2 = -2A$. Thus, $A = -1$.
Setting $x = 2$ in the above expression, we have $2 = 6B$. Thus, $B = \frac{1}{3}$.
Finally, setting $x = -1$ in the above expression, we have $2 = 3C$. Thus, $C = \frac{2}{3}$.
As a result

$$\int \frac{2}{x(x-2)(x+1)}\,dx = \int \left(-\frac{1}{x} + \frac{1}{3}\frac{1}{x-2} + \frac{2}{3}\frac{1}{x+1}\right)\,dx = -\ln|x| + \frac{1}{3}\ln|x-2| + \frac{2}{3}\ln|x+1| + C.$$

(b)

$$\int \frac{x}{(x+1)(x^2+4)}\,dx$$

Solution:

Since $x+1$ is linear and x^2+4 is irreducible quadratic, we have

$$\frac{x}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4) + (Bx+C)(x+1)}{(x+1)(x^2+4)},$$

Thus,

$$x = A(x^2+4) + (x+1)(Bx+C)$$

for some numbers A, B, and C that we need to find.
Setting $x = -1$ in the above expression, we have $-1 = 5A$. Thus, $A = -\frac{1}{5}$. Inserting it into the above equation we find

$$(x+1)(Bx+C) = x + \frac{1}{5}(x^2+4) = \frac{x^2+5x+4}{5} = \frac{(x+4)(x+1)}{5}$$

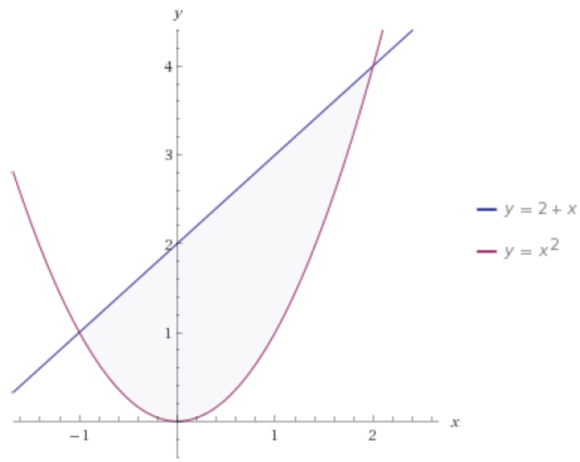
or

$$Bx + C = \frac{x+4}{5},$$

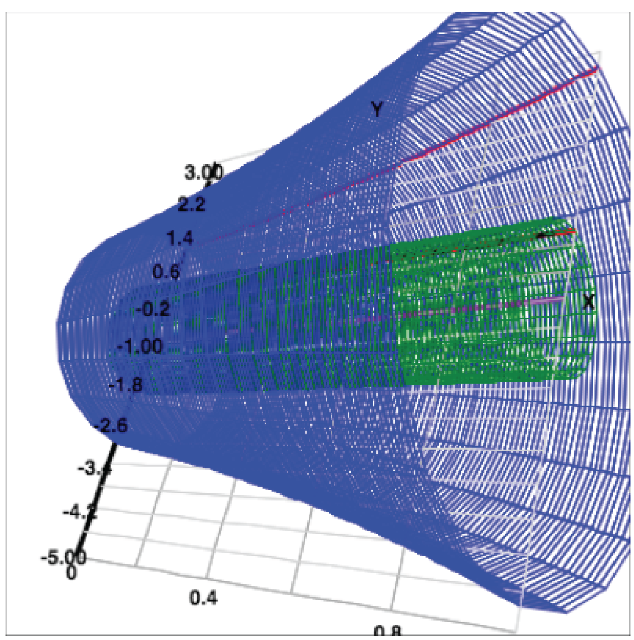
and as a result $B = \frac{1}{5}$ and $C = \frac{4}{5}$. Thus

$$\begin{aligned}
\int \frac{x}{(x+1)(x^2+4)}\,dx &= \int \left(-\frac{1}{5}\frac{1}{x+1} + \frac{1}{5}\frac{x+4}{x^2+4}\right)\,dx \\
&= -\frac{1}{5}\int \frac{dx}{x+1} + \frac{1}{5}\int \frac{x}{x^2+4}\,dx + \frac{4}{5}\int \frac{dx}{x^2+4} \\
&= -\frac{1}{5}\ln|x+1| + \frac{1}{10}\ln(x^2+4) + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C.
\end{aligned}$$

Sketch of the area between curves $y = x + 2$ and $y = x^2$



Sketch of the volume of revolution of $y = e^x$ and $y = 0$ about the axis $y = -1$



7. (10 points) Evaluate the following improper integral

$$\int_1^{\infty} \frac{dx}{(x+1)^2}.$$

Solution:

By the definition

$$\int_1^{\infty} \frac{dx}{(x+1)^2} = \lim_{M\to\infty} \int_1^{M} \frac{dx}{(x+1)^2}.$$

Using the Fundamental Theorem of Calculus

$$\int_1^{M} \frac{dx}{(x+1)^2} = -\frac{1}{x+1}\,|_1^M = -\frac{1}{M+1} + \frac{1}{2}.$$

Since

$$\lim_{M\to\infty} \left(-\frac{1}{M+1} + \frac{1}{2}\right) = \frac{1}{2},$$

we obtain

$$\int_1^{\infty} \frac{dx}{(x+1)^2} = \frac{1}{2}.$$