Practice Final Exam. Solutions.

1. State the domain, range and possible symmetries of the following functions:

   (a)
   $$\sqrt{x^2+1}$$
   Solution: Since $x^2+1 \geq 1$, the domain is $(-\infty, \infty)$, the range is $[1, \infty)$. Since $\sqrt{x^2+1} = \sqrt{(-x)^2+1}$ the function is even.

   (b)
   $$\sqrt{x+1}$$
   Solution: Since the domain for $\sqrt{x}$ is $x \geq 0$, it means that the domain for $\sqrt{x+1}$ is $x \geq -1$. The range is $[0, \infty)$ and there is not symmetry.

   (c)
   $$\frac{x+1}{x-1}$$
   Solution: The domain is all real numbers except $x = 1$. The range is all real numbers except $y = 0$. There is no symmetry.

2. Find the vertical and horizontal asymptotes of the following functions:

   (a)
   $$\frac{\sin x}{\sqrt{x}}$$
   Solution: Since by L'Hopital's rule,
   $$\lim_{x\to 0} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0} 2\sqrt{x}\cos x = 2,$$
   there is not vertical asymptotes. On the other hand by the Sandwich theorem
   $$\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} = 0.$$
   Thus $y = 0$ is the horizontal asymptote.

   (b)
   $$\frac{x+1}{x-1}$$
   Solution: Since,
   $$\lim_{x\to 1^+} \frac{x+1}{x-1} = \infty,$$
   the line $x = 1$ is the vertical asymptote. One could also conclude that from
   $$\lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty.$$
   Since the limit
   $$\lim_{x\to\infty} \frac{x+1}{x-1} = 1,$$
   the line $y = 1$ is the horizontal asymptote.

3. Find the equation of the tangent line at point $(2, 0)$ for $e^y + x^2 = 5$.

Solution: Using the implicit differentiation we have

$$e^y y'(x) + 2x = 0 \quad \Longrightarrow \quad y'(x) = -\frac{2x}{e^y}.$$

Thus the slope at point $(2, 0)$ is $m = -\frac{2\cdot 2}{e^0} = -4$ and the equation of the tangent line is

$$y = -4(x-2) = -4x + 8.$$

4. Find the linear approximation of $\sin x$ at point $\pi/4$.

Solution: Using the formula the linear approximation

$$L(x) = f(a) + f'(a)(x-a),$$

we have with $f(x) = \sin x$, $f'(x) = \cos x$, and $a = \pi/4$,

$$L(x) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}(x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}).$$

5. Evaluate the following limits:

(a)

$$\lim_{x\to 0} \frac{3x}{\sin(4x)}$$

Solution: The limit is in the indeterminate form $\frac{0}{0}$. Thus by the L'Hopital's Rule

$$\lim_{x\to 0} \frac{3x}{\sin(4x)} = \lim_{x\to 0} \frac{3}{4\cos(4x)} = \frac{3}{4}.$$

(b)

$$\lim_{x\to\infty} \frac{x^2+2}{3x^2-4x+5}$$

Solution: The limit is in the indeterminate form $\frac{\infty}{\infty}$. Thus using the L'Hopital's Rule twice, we find

$$\lim_{x\to\infty} \frac{x^2+2}{3x^2-4x+5} = \lim_{x\to\infty} \frac{2x}{6x-4} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}.$$

(c)

$$\lim_{x\to 0^+} x^x$$

Solution: The limit is in the indeterminate form 0^0. Using the logarithm, we have

$$x^x = e^{x\ln x}.$$

Since by the L'Hopital's Rule

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = -\lim_{x\to 0^+} x = 0,$$

we have

$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1.$$

6. Find the absolute maximum and absolute minimum of the function $f(x) = x^3 - x + 1$ on the interval $[0, 1]$. **Solution:** Since $f'(x) = 3x^2 - 1$. We have

$$f'(x) = 3x^2 - 1 = 0 \quad \Longrightarrow \quad x = \pm\frac{1}{\sqrt{3}}.$$

Since our interval is $[0, 1]$, $x = \frac{1}{\sqrt{3}}$ is the only critical point. Thus,

x	0	$\frac{1}{\sqrt{3}}$	1
$f(x)$	1	$\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} + 1$	1

Since $0 < \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} + 1 = \frac{3\sqrt{3}-2}{3\sqrt{3}} < 1$, we have the absolute maximum is 1 and the absolute minimum is $\frac{3\sqrt{3}-2}{3\sqrt{3}}$.

7. The Riemann sum for a function $f(x)$ on the interval $[a, b]$ for an arbitrary n is

$$\sum_{k=1}^{n} f(\bar{x}_k)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.
For the left Riemann sum

$$\bar{x}_k = a + (k-1)\Delta x$$

and for the right Riemann sum

$$\bar{x}_k = a + k\Delta x.$$

Write the left and right Riemann sums for the function $f(x) = \frac{4}{x}$ on the interval $[2, 4]$ for $n = 4$. What can you say about $\int_2^4 f(x)dx$?

Solution: First we find that $\Delta x = \frac{4-2}{4} = \frac{1}{2}$. Thus the right Riemann sum is

$$\sum_{k=1}^{4} f(2+k/2)\frac{1}{2} = \frac{1}{2}\left(f(5/2) + f(3) + f(7/2) + f(4)\right) = \frac{1}{2}\left(\frac{4\cdot 2}{5} + \frac{4}{3} + \frac{4\cdot 2}{7} + \frac{4}{4}\right) = 2\left(\frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right).$$

Hence the left Riemann sum is

$$\sum_{k=0}^{3} f(2+k/2)\frac{1}{2} = \frac{1}{2}\left(f(2) + f(5/2) + f(3) + f(7/2)\right) = \frac{1}{2}\left(\frac{4}{2} + \frac{4\cdot 2}{5} + \frac{4}{3} + \frac{4\cdot 2}{7}\right) = 2\left(\frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7}\right).$$

Since the function $f(x) = \frac{4}{x}$ is monotonically decreasing function, the left Reimann sum is overestimate and the right Riemann sum is underestimate, i.e.

$$2\left(\frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right) \le \int_2^4 f(x)dx \le 2\left(\frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7}\right).$$

8. Using the Fundamental Theorem of Calculus find the following derivatives:

   (a)

$$\frac{d}{dx}\int_1^x t^2 dt$$

   Solution: By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_1^x t^2 dt = x^2.$$

(b)

$$\frac{d}{dx}\int_{x^2}^{2} t^2 dt$$

Solution: By the Fundamental Theorem of Calculus and the chain rule

$$\frac{d}{dx}\int_{x^2}^{2} t^2 dt = -(x^2)^2 \cdot 2x = -2x^5.$$

9. Find the following antiderivatives:

(a)

$$\int (x^2 - 1)dx$$

Solution:

$$\int (x^2 - 1)dx = \frac{x^3}{3} - x + C.$$

(b)

$$\int \frac{dx}{2x^2+1} dx$$

hint: $\int \frac{dx}{x^2+1} dx = \tan^{-1} x + C$

Solution:

$$\int \frac{dx}{2x^2+1} dx = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}x) + C.$$

(c)

$$\int (x^2+2)^2 dx$$

Solution:

$$\int (x^2+2)^2 dx = \int (x^4 + 4x^2 + 4)dx = \frac{x^5}{5} + \frac{4x^3}{3} + 4x + C.$$

10. Using the Fundamental Theorem of Calculus compute the following integrals:

(a)

$$\int_1^2 x^3 dx$$

Solution:

$$\int_1^2 x^3 dx = \left.\frac{x^4}{4}\right|_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

(b)

$$\int_0^{e-1} \frac{dx}{x+1} dx$$

Solution:

$$\int_0^{e-1} \frac{dx}{x+1} dx = \ln(x+1)|_0^{e-1} = \ln e - \ln 1 = 1.$$