MATH 1131 – Calculus I

Practice Exam 3. Solutions.

1. Find the linear approximation of $\sqrt[4]{x}$ at point $a = 2$ and using it estimate $\sqrt[4]{16.04}$.

Solution: The formula for the linear approximation is

$$L(x) = f(16) + f'(16)(x - 16),$$

where $f(x) = \sqrt[4]{x}$. Thus $f(16) = \sqrt[4]{16} = 2$ and $f'(16) = \frac{1}{4}16^{-3/4} = \frac{1}{4\cdot 8} = \frac{1}{32}$. Hence

$$L(x) = 2 + \frac{1}{32}(x - 16),$$

and as a result

$$\sqrt[4]{16.04} \approx L(16.04) = 2 + \frac{1}{32}(16.04 - 16) = 2\frac{1}{800}.$$

2. Evaluate the following limits:

(a)

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin(2x)}$$

Solution: Since $\tan 0 = \sin 0 = 0$, the limit is in the indeterminate form $\frac{0}{0}$. Thus be the L'Hopital's Rule

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{3\sec^2(3x)}{2\cos(2x)} = \frac{3}{2}.$$

(b)

$$\lim_{x\to\infty} \frac{e^{x^2}}{x^2}$$

Solution: The limit is in the indeterminate form $\frac{\infty}{\infty}$. Thus be the L'Hopital's Rule

$$\lim_{x\to\infty} \frac{e^{x^2}}{x^2} = \lim_{x\to\infty} \frac{2xe^{x^2}}{2x} = \lim_{x\to\infty} e^{x^2} = \infty.$$

(c)

$$\lim_{x\to\infty} \left(1 + \frac{1}{2x}\right)^x$$

Solution: The limit is in the indeterminate form 1^∞. Using logarithm, we have

$$\left(1 + \frac{1}{2x}\right)^x = e^{x\ln\left(1+\frac{1}{2x}\right)}.$$

The limit

$$\lim_{x\to\infty} x\ln\left(1 + \frac{1}{2x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{2x}\right)}{\frac{1}{x}}$$

is in indeterminate form $\frac{\infty}{\infty}$. Using the L'Hopital's Rule

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{2x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{2x}{2x+1}\cdot\frac{-1}{2x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x}{2x+1} = \frac{1}{2}.$$

Hence the answer is $e^{1/2} = \sqrt{e}$.

3. Find the minimum value of $5a + 3b$ for $a > 0$ and $b > 0$, such that $ab = 25$.

Solution: From $ab = 25$ we have $a = 25/b$. Hence we need to maximize $125/b + 3b$, which is a function of b. Taking derivative we have $-125/b^2 + 3$. The critical point is when $-125/b^2 + 3 = 0$ or $b^2 = \frac{125}{3}$ or $b = \frac{5\sqrt{5}}{\sqrt{3}}$, hence $a = \frac{25\sqrt{3}}{5\sqrt{5}} = \sqrt{15}$. Since the second derivative is $250/b^3 > 0$, the minimum is

$$\frac{5\sqrt{5}}{\sqrt{3}} + \sqrt{15}.$$

4. Using the following graphing guidelines sketch the function $f(x) = x \ln^2 x$:

(a) What's the domain of $f(x)$?
Solution:
Because the domain of $\ln x$, $x > 0$, the domain of $f(x)$ is also $x > 0$ or $(0, \infty)$.

(b) Is $f(x)$ even or odd?
Solution:
Neither.

(c) Find $f'(x)$.
Solution:
Using the product and chain rules

$$f'(x) = \ln^2 x + 2\ln x.$$

(d) Find $f''(x)$.
Solution:
Using the chain rule

$$f''(x) = \frac{2\ln x}{x} + \frac{2}{x}.$$

(e) Find the critical points (i.e. where $f'(x) = 0$).
Solution:

$$f'(x) = \ln^2 x + 2\ln x = \ln x(\ln x + 2) = 0 \quad \Longrightarrow \quad \ln x = 0 \quad \text{or} \quad \ln x + 2 = 0.$$

Thus $x = 1$ or $x = e^{-2}$.

(f) Find inflection points (i.e. where $f''(x) = 0$).
Solution:

$$f''(x) = 0 \quad \Longrightarrow \quad 2\ln x + 2 = 0 \quad \Longrightarrow \quad x = e^{-1}.$$

(g) Find the intervals on which the function is increasing, decreasing.
Solution:
Since $f' > 0$ on the intervals $(0, e^{-2})$ and $(1, \infty)$, $f(x)$ is increasing on $(0, e^{-2})$ and $(1, \infty)$. Consequently, $f' < 0$ on $(e^{-2}, 1)$ and as a result $f(x)$ is decreasing on $(e^{-2}, 1)$.

(h) Find the intervals on which the function is concave up, concave down.
Solution:
Since $f'' > 0$ on (e^{-1}, ∞), $f(x)$ is concave up on (e^{-1}, ∞). Consequently, $f'' < 0$ on $(0, e^{-1})$ and as a result $f(x)$ is concave down on $(0, e^{-1})$.

(i) Identify extreme points.
Solution:
$x = e^{-2}$ is local maximum, $x = 1$ is local minimum, $x = e^{-1}$ is an inflection point.

(j) Locate vertical asymptotes

Solution:

There is no vertical asymptotes. Notice that

$$\lim_{x\to 0^+} x\ln^2 x = 0.$$

(k) Locate horizontal asymptotes

Solution:

There is no horizontal asymptotes,

$$\lim_{x\to\infty} x\ln^2 x = \infty.$$

(l) Find x-intercepts (i.e. where $f(x) = 0$).

Solution:

$$f(x) = 0 \implies x\ln^2 x = 0 \implies \ln x = 0 \implies x = 1.$$

(m) Find y-intercept (i.e. when $f(0)$).

Solution:

$x = 0$ is not in the domain of $f(x)$, but $\lim_{x\to 0^+} x\ln^2 x = 0$.

(n) Sketch the graph

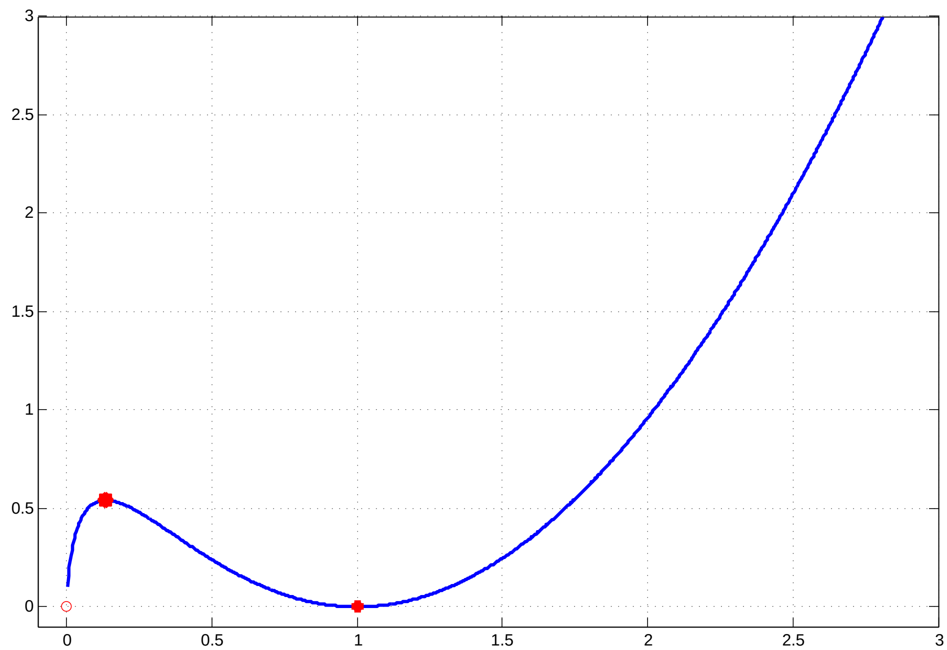
Solution:

See next page.

5. Find the interval on which Rolle's Theorem applied to $f(x) = x^2(x+3)$. Find a point c in that interval at which $f'(c) = 0$.

Solution: Since $f(0) = f(-3) = 0$ we can take the interval $[-3, 0]$. The function is a polynomial hence it is continuous and differentiable everywhere and the conditions of the Rolle's Theorem are satisfied. The Rolle's Theorem guarantees a point c such that $f'(c) = 0$.

Computing $f'(x) = 3x^2 + 6x$. Thus $f'(c) = 0$ implies $3c^2 + 6c = 0$. Thus $c = 0$ or $c = -2$. The point $c = -2 \in (-3, 0)$.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