MATH 1131 – Calculus I

Practice Exam 2. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. Find the equation of the tangent line at point $(1, 1)$ for $y^3 + x^2 = 2$.

   Solution: Assume that y is a function of x, i.e. $y = y(x)$. Differentiating both sides of the equation and using the chain rule we obtain

$$3y^2y' + 2x = 0 \quad \Longrightarrow \quad y' = -\frac{2x}{3y^2}.$$

   Thus the slope at the point $(1, 1)$ is $-2/3$ and the equation of the tangent line is

$$y - 1 = -\frac{2}{3}(x - 1) \quad \Longrightarrow \quad y = -\frac{2}{3}x + \frac{5}{3}.$$

2. Evaluate the following limits:

   (a)

$$\lim_{x\to 0} \frac{\tan(3x)}{5x}$$

   Solution: Using $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and as a result $\lim_{x\to 0} \frac{\sin(3x)}{x} = 3$, we have

$$\lim_{x\to 0} \frac{\tan(3x)}{5x} = \frac{1}{5}\lim_{x\to 0} \frac{\sin(3x)}{x \cdot \cos(3x)} = \frac{1}{5}\lim_{x\to 0} \frac{\sin(3x)}{x} \lim_{x\to 0} \frac{1}{\cos(3x)} = \frac{1}{5} \cdot 3 \cdot \frac{1}{\cos 0} = \frac{3}{5}.$$

   (b)

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin x}$$

   Solution: Again using $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{\sin(5x)}{x} = 5$, we have

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin x} = \lim_{x\to 0} \frac{x \cdot \sin(5x)}{x \cdot \sin x} = \lim_{x\to 0} \frac{\sin(5x)}{x} \lim_{x\to 0} \frac{x}{\sin x} = 5 \cdot 1 = 5.$$

   (c)

$$\lim_{x\to 0} \frac{(\cos x - 1)(x + 1)}{2x}$$

   Solution: Using $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$, we have

$$\lim_{x\to 0} \frac{(\cos x - 1)(x + 1)}{2x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \lim_{x\to 0} \frac{x + 1}{2} = 0 \cdot \frac{1}{2} = 0.$$

   (d)

$$\lim_{x\to 0} \frac{e^x - 1}{2x}$$

   Solution: Using $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, we have

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{e^x - 1}{x} = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

3. Use logarithmic differentiation to compute $f'(x)$ for

$$f(x) = \frac{\sqrt[4]{x^2+1}}{(x^2+10)\sqrt[3]{x^3+1}}$$

Solution: Taking logarithm of both sides we have

$$\ln f(x) = \frac{1}{4}\ln(x^2+1) - \ln(x^2+10) - \frac{1}{3}\ln(x^3+1).$$

Differentiating both sides with respect of x and using the chain rule we obtain

$$\frac{f'(x)}{f(x)} = \frac{1}{4}\frac{2x}{x^2+1} - \frac{2x}{x^2+10} - \frac{1}{3}\frac{3x^2}{x^3+1}.$$

Hence

$$f'(x) = \left(\frac{1}{2}\frac{x}{x^2+1} - \frac{2x}{x^2+10} - \frac{x^2}{x^3+1}\right)\frac{\sqrt[4]{x^2+1}}{(x^2+10)\sqrt[3]{x^3+1}}.$$

4. Compute $f'(x)$ for

(a)

$$f(x) = (x^3-10x)^3(x^2+1)^2$$

Solution: Applying the product and chain rules, we have

$$\begin{aligned} f'(x) &= 3(x^3-10x)^2(3x^2-10)(x^2+1)^2 + (x^3-10x)^3 2(x^2+1)2x \\ &= (x^3-10x)^2(x^2+1)\left[3(3x^2-10)(x^2+1) + 4x(x^3-10x)\right]. \end{aligned}$$

(b)

$$f(x) = \frac{\sqrt{x}+1}{x+1}$$

Solution: Applying the quotient rules, we have

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+1) - (\sqrt{x}+1)}{(x+1)^2} = \frac{-\frac{\sqrt{x}}{2} + \frac{1}{2\sqrt{x}} - 1}{(x+1)^2}.$$

(c)

$$f(x) = \tan^{-1}(x^2)$$

Solution: Using the rules for inverse trigonometric functions and the chain rule we obtain

$$f'(x) = \frac{1}{(x^2)^2+1}2x = \frac{2x}{x^4+1}.$$

(d)

$$f(x) = \sin^{-1}(x^2+1)$$

Solution: Again using the rules for inverse trigonometric functions and the chain rule we obtain

$$f'(x) = \frac{2x}{\sqrt{1-(x^2+1)^2}}.$$

(e)

$$f(x) = x^{x^2}$$

Solution: First, using the properties of exponential function and the logarithm we have

$$f(x) = x^{x^2} = e^{\ln x^{x^2}} = e^{x^2\cdot\ln x}.$$

Using the chain and product rules we obtain

$$f'(x) = e^{x^2\ln x}(2x\cdot\ln x + x) = x^{x^2}(2x\cdot\ln x + x).$$

5. A square initially has dimensions 2 in by 2 in. All sides begin increasing in length at a rate of 1 in/sec. At what rate is the area of the square increasing after 10 sec?

Solution: Let $d(t)$ denote the side of the square at time t. Thus $d(t)$ is a function of time and in particular $d(0) = 2$ in. The area of the square $A(t) = d^2(t)$ (again a function of time). Differentiating with respect to time and using the chain rule we have

$$A'(t) = 2 \cdot d(t) \cdot d'(t) \quad \Longrightarrow \quad A'(10) = 2 \cdot d(10) \cdot d'(10).$$

Since $d'(t) = 1$ in/sec. for any t and after 10 seconds the side is 12 inches. It means $d(10) = 12$ and as a result $A'(10) = 2 \cdot 12 \cdot 1 = 24\ in^2/sec.$