MATH 1131 – Calculus I

Practice Exam 1. Solutions.

No calculators. Show your work. Clearly mark each answer.

1. State the domain and the range of the function $\sqrt{1-x^2}$. Is this function one-to-one? Sketch the graph.

   Solution: Since the domain for $\sqrt{x}$ is $x \geq$, the domain for $\sqrt{1-x^2}$ is the set of all x such that $1-x^2 \geq 0$. This inequality is equivalent to $-1 \leq x \leq 1$. Thus the domain $D = [-1,1]$.

   To obtain the range, we notice that the largest value for $\sqrt{1-x^2}$ is 1 when $x = 0$ and the smallest value is 0 when $x = 1$ or $x = -1$. Thus the range $R = [0,1]$. This function is obviously is not one-to-one since the function is 0 for $x = 1$ and $x = -1$. The graph of this function is the upper half of the unit circle.

2. For which value of a is the following function continuous?

$$f(x) = \begin{cases} x^2 + a, & x > 0 \\ 5x - 3, & x \leq 0. \end{cases}$$

   Solution: Since both pieces of the function $f(x)$, $x^2 + a$ and $5x - 3$ are continuous, $f(x)$ is continuous if

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x).$$

   This is equivalent to

$$\lim_{x \to 0^-} x^2 + a = a = \lim_{x \to 0^+} 5x - 3 = -3.$$

   Which means that in order for $f(x)$ to be continuous we must have $x = -3$.

3. Find the following limits if they exist.

   (a)

$$\lim_{x \to 1^+} \frac{3x-2}{x-1}$$

   Solution: Since $3 \cdot 1 - 2 = 1 > 0$ and since we approach 1 from the right

$$\lim_{x \to 1^+} \frac{3x-2}{x-1} = +\infty.$$

   (b)

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$$

   Solution: Multiplying and dividing by the conjugate, we obtain

$$\frac{\sqrt{x+1}-1}{x} = \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}.$$

   Thus

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}.$$

(c)

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(x)}$$

Solution: Using the double angle formula $\sin(2x) = 2\sin x\cos x$, we obtain

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(x)} = \lim_{x\to 0}\frac{2\sin x\cos x}{\sin x} = 2\lim_{x\to 0}\cos x = 2\cos 0 = 2.$$

(d)

$$\lim_{x\to\infty}\frac{\sqrt{x^4+x+1}-x^2}{x^2}$$

Solution: Multiplying and dividing by the conjugate, we obtain

$$\frac{\sqrt{x^4+x+1}-x^2}{x^2} = \frac{(\sqrt{x^4+x+1}-x^2)(\sqrt{x^4+x+1}+x^2)}{x^2(\sqrt{x^4+x+1}+x^2)} = \frac{x^4+x+1-x^4}{x^2(\sqrt{x^4+x+1}+x^2)} = \frac{x+1}{x^2(\sqrt{x^4+x+1}+x^2)}$$

Dividing by x^2 we obtain

$$\lim_{x\to\infty}\frac{\sqrt{x^4+x+1}-x^2}{x^2} = \lim_{x\to\infty}\frac{x+1}{x^2(\sqrt{x^4+x+1}+x^2)} = \lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{\sqrt{x^4+x+1}+x^2} = 0.$$

4. Find the vertical asymptotes of the function $e^{-\frac{1}{x}}$.

   Solution: Since

$$\lim_{x\to 0-} -\frac{1}{x} = +\infty,$$

   we have

$$\lim_{x\to 0-} e^{-\frac{1}{x}} = e^{\infty} = \infty.$$

   Thus $x = 0$ is a vertical asymptote.

5. Find the equation of the line passing through the points $(-2,-3)$ and $(1,1)$.

   Solution: First we compute the slope

$$m = \frac{-3-1}{-2-1} = \frac{4}{3}.$$

   Thus the equation of the line is

$$y - 1 = \frac{4}{3}(x-1) \quad \text{or} \quad y = \frac{4}{3}x - \frac{1}{3}.$$

6. Suppose a stone is thrown vertically upward from a height of 4 feet. The height s in feet of the stone above the ground t seconds after it is thrown is

$$s(t) = -16t^2 + 32t + 4.$$

   (a) What is average velocity of the stone after 1 second?

   Solution:

$$Av = \frac{s(1)-s(0)}{1-0} = -16+32+4-4 = 16.$$

(b) What is average velocity of the stone after t seconds?

Solution:

$$Av = \frac{s(t) - s(0)}{t - 0} = \frac{-16t^2 + 32t + 4 - 4}{t} = -16t + 32.$$

(c) What is initial velocity of the stone?

Solution: Initial velocity would be the instantaneous velocity, which is equal to

$$v(0) = \lim_{t \to 0} \frac{s(t) - s(0)}{t - 0} = \lim_{t \to 0} -16t + 32 = 32.$$

(d) When does the stone strike the ground?

Solution: The stone strikes the ground when $s = 0$. Solving the equation $-16t^2 + 32t + 4 = 0$ using the quadratic formula and keeping in mind that the solution should be positive, we obtain $t = \frac{8+\sqrt{80}}{8}$. Or after $1 + \frac{\sqrt{5}}{2}$ seconds.