Minimal Pair Constructions and Iterated Trees of Strategies

Dedicated to Anil Nerode on the occasion of his sixtieth birthday

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0. Introduction. We use the iterated trees of strategies approach developed in [LL1], [LL2] to prove some theorems about minimal pairs. In Sections 1-3, we show how to use these methods to prove the Minimal Pair Theorem of Lachlan [L] and Yates [Y]:

Theorem 3.4 (Minimal Pair): There exist nonrecursive r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

In Section 4, we add requirements to prove:

Theorem 4.3: There are r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}', \mathbf{b}' > \mathbf{0}'$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

And in Section 5, we construct a minimal pair of r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}'$ and $\mathbf{b}'$ form a minimal pair over $\mathbf{0}'$, i.e.,

Theorem 5.4: There are r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' | \mathbf{b}'$, $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, and $\mathbf{a}' \wedge \mathbf{b}' = \mathbf{0}'$.

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$\mathbb{N}$ is the set of natural numbers. Given a set P, we let $|P|$ denote the cardinality of P.

A *string* is a finite sequence of letters from an alphabet. If S is an alphabet, we let $S^{<\omega}$ be the set of all strings from S. We write $\sigma \subset \tau$ if τ properly extends σ, and $\sigma|\tau$ if σ and τ are incomparable. For $\sigma,\tau \in S^{<\omega}$, we let $lh(\sigma)$ denote the cardinality of the domain of σ. If $\sigma \neq \langle\ \rangle$ (the empty string), then σ^- is the unique $\tau \subset \sigma$ such that $lh(\sigma) = lh(\tau)+1$. We define the string $\sigma\hat{}\tau$ by

$$\sigma\hat{}\tau(x) = \begin{cases} \sigma(x) & \text{if } x < lh(\sigma) \\ \tau(x-lh(\sigma)) & \text{if } lh(\sigma) \leq x < lh(\sigma)+lh(\tau). \end{cases}$$

If $x \leq lh(\sigma)$, then $\sigma|x$, *the restriction of* σ *to x,* is the string τ of length x such that $\tau(y) = \sigma(y)$ for all $y < x$. Restriction is similarly defined for infinite sequences from an alphabet. We use interval notation for strings. Thus $[\sigma,\tau] = \{\rho: \sigma \subseteq \rho \subseteq \tau\}$. $\sigma\wedge\tau$ denotes the longest ρ such that $\rho \subseteq \sigma,\tau$.

A *tree* is a set of strings which is closed under restriction. The *paths* through a tree T are the infinite sequences Λ such that $\Lambda|x \in T$ for all $x \in \mathbb{N}$. We let $[T]$ denote the set of paths through T.

The *high/low hierarchy* for $\mathcal{R}$ is defined as follows. For $n \geq 1$, we say that $\mathbf{a}$ is low_n ($\mathbf{a} \in \mathbf{L_n}$) if $\mathbf{a}^{(\mathbf{n})} = 0^{(\mathbf{n})}$, and $\mathbf{a}$ is $high_n$ ($\mathbf{a} \in \mathbf{H_n}$) if $\mathbf{a}^{(\mathbf{n})} = 0^{(\mathbf{n+1})}$. If $0^{(\mathbf{n})} < \mathbf{a}^{(\mathbf{n})} < 0^{(\mathbf{n+1})}$ for all n, then we say that $\mathbf{a}$ is *intermediate*.

$\langle\Phi^k_e: e \in \mathbb{N}\rangle$ will be the standard enumeration of all partial recursive functionals of k variables. (We will frequently suppress the superscript, writing Φ_e for Φ^k_e.) Thus $\Phi^k_e(A;x_1,\cdots,x_k) = y$ if the eth partial recursive functional of k variables, computing from oracle A and input $x_1,\cdots,x_k$, outputs the value y. For each $e,k \in \mathbb{N}$, we will have a recursive approximation $\langle\Phi^k_{e,s}: s \in \mathbb{N}\rangle$ to Φ^k_e. We say that $\Phi^k_{e,s}(A;x_1,\ldots,x_k)\downarrow$ if we obtain an output from this computation in fewer than s steps; otherwise, $\Phi^k_{e,s}(A;x_1,\ldots,x_k)\uparrow$. If $\Phi^k_{e,s}(A;x_1,\ldots,x_k)\downarrow$, then we let the *use* of this computation be 1+u, where u is the greatest element v for which a question "$v \in A$?" is asked of the A oracle during the computation. We will also be constructing partial recursive functionals. We define these functionals within a recursive construction by *declaring axioms* $\Delta(\sigma;\overline{x}) = y$ to reflect the fact that the partial recursive functional Δ with input $\overline{x}$ produces output y when computing from any oracle $A \supset \sigma$. Other notation follows [So].

1. The Basic Modules for a Minimal Pair. We construct r.e. sets A_0 and A_1 such that neither is recursive, and any set recursive in both must be recursive. The requirements which we satisfy are the standard ones. In this section, we present the basic modules for the two types of requirements. We will have three trees of strategies, T^0, T^1, and T^2, and relate the action taken for the basic modules to the nodes of the trees on which we act for these modules.

The Requirements: We let $R^{0,1}_{e,i}$ be the requirement stating that $A_i \neq \Phi_e(\emptyset)$. This requirement is said to have *type 0* and *dimension 1.* (Types are used to differentiate between the various kinds of requirements. Dimensions indicate the trees on which the construction first begins to act, i.e., the complexity level of the requirement.)

We let $R^{1,2}_e$ be the requirement stating that if $\Phi_e(A_0) = \Phi_e(A_1)$ is total, then $\Phi_e(A_0)$ is recursive. This requirement has *type 1* and *dimension 2.* We recall that by Posner's Lemma [P], it suffices to satisfy these requirements, as given any n and m, there is an e such that $\Phi_n(A_0) = \Phi_e(A_0)$ and $\Phi_m(A_1) = \Phi_e(A_1)$.

Fix a recursive ordering $\{R_i: i \in \mathbb{N}\}$ of all requirements. We say that R_i *has higher priority than* R_j if $i < j$. ■

The Basic Module: *Nonrecursiveness requirements.* We satisfy the requirement $R^{0,1}_{e,i}$ by constructing a partial recursive functional Δ_i which is total on oracle A_i such that for some x, if $\Phi_e(\emptyset;x)\downarrow$, then $\Phi_e(\emptyset;x) \neq \Delta_i(A_i;x)$. Each instance of this requirement on T^1 will be associated with a different argument x, and each instance on T^0 will be associated with a *stage* s as well. At stage s, we *declare* an axiom $\Delta_{i,s}(A^s_i;x) = m$ with use x+1, setting m = 0 unless $\Phi_{e,s}(\emptyset;x)\downarrow = 0$, in which case we set m = 1. If, at some stage $t \geq s$ which deals with the T^1-module, we find that $\Phi_{e,t}(\emptyset;x)\downarrow = 0$, then we place x into A^t_i. If $t > s$, this will allow us to redefine $\Delta_{i,t}(A^t_i;x) = 1$, with use t+1.

There are two possible types of outcomes. If, during the construction, we never see a computation $\Phi_{e,t}(\emptyset;x)\downarrow = 0$, then $\Delta_i(A_i;x) = 0$ and we will ensure that either $\Phi_e(\emptyset;x)\uparrow$, or $\Phi_e(\emptyset;x)\downarrow \neq 0$ by protecting a computation, if necessary. This outcome will be called the *finite outcome* on the tree T^0 and the *infinite outcome* on the tree T^1. If we see such a computation, then we place x into A_i unless this happens the first time we encounter the requirement, and ensure that $\Phi_e(\emptyset;x)\downarrow = 0 \neq 1 = \Delta_i(A_i;x)$. This outcome will be called the *infinite outcome* on T^0 and the *finite outcome* on T^1. ■

The Basic Module: *Minimal pair requirements.* The basic module for $R_e^{1,2}$ will be a finite tree consisting of two comparable nodes, of types (1,0) and (1,1), essentially following the Yates minimal pair construction [Y]. The first node checks to see whether $\Phi_e(A_0)$ and $\Phi_e(A_1)$ are both total, and the second tries to find an x such that $\Phi_e(A_0;x) \neq \Phi_e(A_1;x)$. If it fails to find such an x, it defines a total recursive function $\Xi = \Phi_e(A_0)$.

There will be one finite tree for each such requirement along each infinite path through the top tree T^2 of our tree of strategies. On the next tree down, T^1, we will work with similar finite trees, each associated with a given x. And on the recursive tree T^0 where the construction takes place, the modules derived from those on T^1 will also be associated with a stage s. Action on T^0 for the module associated with x and s is as follows. The first node, of type (1,0), asks if $\Phi_{e,s}(A_0^s;x)\downarrow$ and $\Phi_{e,s}(A_1^s;x)\downarrow$. If the answer is *no*, then we follow the finite outcome on T^0. (We will now have to deal with other T^0-modules derived from this T^1-module, so make infinitely many attempts to find a *yes* answer to this question. If one is never found, then either $\Phi_e(A_0)$ is not total, or $\Phi_e(A_1)$ is not total, so the requirement is satisfied.) If the answer is *yes*, then we follow the infinite outcome on T^0, and action to be taken will be determined by the second node of the module. The second node of the module has type (1,1) and asks whether $\Phi_{e,s}(A_0^s;x) \neq \Phi_{e,s}(A_1^s;x)$. If the answer is *yes,* then we follow the finite outcome on T^0, and will preserve this disagreement as long as the node is derived from a node on the true path of T^1. If the answer is *no,* then we follow the infinite outcome on T^0, and *declare* an axiom $\Xi_s(\emptyset;x) = \Phi_{e,s}(A_0^s;x)$. We show that, in this case, for all $t \geq s$, either $\Phi_{e,t}(A_0^t;x) = \Xi(\emptyset;x)$ or $\Phi_{e,t}(A_1^t;x) = \Xi(\emptyset;x)$. Hence if we encounter such a requirement for every x on T^1 and $\Phi_e(A_0;x)\downarrow = \Phi_e(A_1;x)\downarrow$ for all x, then $\Xi(\emptyset;x) = \Phi_e(A_0;x)$ for all x. ■

2. Systems of trees. The construction of a minimal pair will use systems of trees. We refer the reader to [LL2] for a more general development of such systems. As we will prove theorems in this paper requiring starting trees at several different levels, we fix $n \in \mathbb{N}$, and indicate how to construct T^i for $i \leq n$.

Definition 2.1 (Definition of trees): We set $T^{-1} = \{0,\infty\}$ and $T^0 = \{0,\infty\}^{<\omega}$. If $0 < k \leq n$ and T^{k-1} has been defined, let

$$T^k = \{\sigma \in (T^{k-1})^{<\omega}: \forall i<lh(\sigma)\forall j<lh(\sigma)(i \neq j \rightarrow \sigma(i) \neq \sigma(j))\}.$$

$\mathcal{T}^k = \langle T^k,\subseteq\rangle$ is the *kth tree of strategies,* ordered by inclusion. We refer to the elements of T^k as *nodes* of T^k, and view each node of T^k as following its immediate predecessor by a designated node of T^{k-1}. If $\sigma \in T^k$, $\xi \in T^{k-1}$, and $\sigma = \sigma^{-}\hat{}\langle\xi\rangle$, then we call ξ the *outcome for* σ^- *along* σ, and define $\mathrm{out}(\sigma) = \xi$. If $j \leq k$, then we define $\mathrm{out}^j(\sigma)$ by reverse induction; $\mathrm{out}^k(\sigma) = \sigma$, and $\mathrm{out}^{j-1}(\sigma) = \mathrm{out}(\mathrm{out}^j(\sigma))$. If $\sigma \subseteq \tau \in T^k$ and $\mathrm{lh}(\sigma) > 0$, then we say that σ^- *has finite (infinite, resp.) outcome along* τ if either $k = 0$ and $\mathrm{out}(\sigma) = 0$ ($\mathrm{out}(\sigma) = \infty$, resp.), or $k > 0$ and $\mathrm{out}(\sigma)^-$ has infinite (finite, resp.) outcome along $\mathrm{out}(\sigma)$. ■

We will need a one-to-one *weight* function on elements of $T = \cup\{T^k: 0 \leq k \leq n\}$ which will ω-order T. (We take the disjoint union here, differentiating between the empty nodes of the various trees.) Weights will provide automatic initialization to protect computations from injury. Condition (2.3) below will be used to show that the path generating function, λ, introduced below, respects weights (see condition (2.11)).

Definition 2.2: It is routine to check that a one-to-one recursive *weight function* $\mathrm{wt}:T \to \mathbb{N}$ can be defined to satisfy the following properties for all $\sigma,\tau,\rho \in T^k$.

(2.1) If $\sigma \subset \tau$ then $\mathrm{wt}(\sigma) < \mathrm{wt}(\tau)$.

(2.2) If $\mathrm{out}(\sigma) \in T$, then $\mathrm{wt}(\mathrm{out}(\sigma)) < \mathrm{wt}(\sigma)$.

(2.3) If $\sigma \neq \tau$ and $\mathrm{out}(\rho) \subset \mathrm{out}(\tau)$ for all $\rho \subseteq \sigma$, then $\mathrm{wt}(\sigma) < \mathrm{wt}(\tau)$.

Requirements of dimension r will be assigned to nodes of trees T^k for $k \geq r$, and subrequirements of these requirements will be assigned to nodes of T^k for $k < r$. This assignment of requirements will proceed by induction on n-k, and will depend on definitions introduced by simultaneous induction. In Step 1, we will define the path generating function λ on nodes σ of trees which have already had requirements assigned to all $\tau \subset \sigma$. If $\sigma \in T^k$, then $\lambda(\sigma)$ will be a node on T^{k+1}. Given a path $\Lambda \in [T^k]$, $\{\lambda(\sigma): \sigma \subset \Lambda\}$ gives an approximation to a path $\lambda(\Lambda) \in [T^{k+1}]$. When $\lambda(\sigma)|\lambda(\sigma^-)$, a link will be formed on T^k. These links, defined in Step 2, will prevent action by nodes of T^k which do not seem to come from nodes on higher trees lying on the true path approximation. We will have to decide which nodes of T^{k+1} are eligible to assign subrequirements to a given node of T^k. Conditions ensuring consistency between the different trees enter into this decision, and these conditions are delineated in Step 3. The requirement assignment process is described in Step 4.

Step 1: **Definition of the path generating function λ.** Given a node $\eta \in T^k$ such that requirements have been assigned to all predecessors of η, the function λ will define a node $\lambda(\eta) \in T^{k+1}$. The process is meant to capture the following situation. For each $\xi \subset \eta$, ξ will be derived from a node $\sigma \in T^{k+1}$. A sentence M_σ will implicitly be associated with σ, and a fragment M_ξ of that sentence will be associated with ξ. Suppose that M_σ begins with a universal quantifier. If σ has dimension $\geq$ k+1, we bound the leading block of universal quantifiers by wt(σ) which, by (2.1), increases with lh(ξ). As long as each ξ succeeds in satisfying its sentence M_ξ, the approximation given by λ predicts that $\sigma\hat{}\langle\nu\hat{}\langle\beta\rangle\rangle \subseteq \lambda(\eta)$, where ν will be the initial derivative of σ along η (defined below) and β is the outcome of ν along η. If we find a first ξ for which M_ξ is false, then $\sigma\hat{}\langle\xi\hat{}\langle\beta\rangle\rangle \subseteq \lambda(\sigma)$, where β is the outcome of ξ along η. If M_σ begins with an existential quantifier, then we proceed as above after replacing M_ξ with $\neg M_\xi$. (If dim(σ) $\leq$ k, then we just copy the outcome from T^k to T^{k+1}.)

If $\eta = \langle\ \rangle$ then $\lambda(\eta) = \langle\ \rangle$. Suppose that $\eta \neq \langle\ \rangle$. By (2.4), it will follow by induction that up(η^-) $\subseteq \lambda(\eta^-)$, where up($\eta^-$) is the node of T^{k+1} from which η^- is derived. (up(η^-) has been defined inductively in Step 4 for η^-.)

(2.4) If either up(η^-) = $\lambda(\eta^-)$ or η^- has infinite outcome along η, then we set $\lambda(\eta)$ = up(η^-)$\hat{}\langle\eta\rangle$ and call η^- the *principal derivative of* up(η^-) *along* η. We set $\lambda(\eta) = \lambda(\eta^-)$ otherwise, and define the *principal derivative of* up(η^-) *along* η to be the principal derivative of up(η^-) along η^-.

It follows from (2.4) that:

(2.5) If $\sigma \subseteq \lambda(\eta)$ then out(σ) $\subseteq \eta$ and λ(out(σ)) = σ; and

(2.6) If $\lambda(\eta^-) \supseteq \sigma$ and $\lambda(\eta) \not\supseteq \sigma$, then for all $\delta \supseteq \eta$, $\lambda(\delta) \not\supseteq \sigma$.

We define $\lambda^r(\eta)$ for r $\in$ [k,n] by $\lambda^k(\eta) = \eta$ and $\lambda^r(\eta) = \lambda(\lambda^{r-1}(\eta))$ for r > k. And if r $\geq$ k and $\zeta \in T^r$, we call ξ the *principal derivative of* ζ *along* η if either r = k and $\xi = \zeta$, or r > k and there is a $\sigma \in T^{k+1}$ such that ξ is the principal derivative of σ along η and σ is the principal derivative of ζ along $\lambda(\eta)$.

Step 2: Links. We will place restrictions on the stages of the construction at which nodes are eligible to be *switched* by the approximation to the true path. One restriction requires a node to be *free* when it is switched by the true path approximation, i.e., that it not be contained in any *link*. Links are formed when a switch occurs, and can be broken when the outcome of a switched node is switched back. (Links correspond to initialization, after injury, in the standard approach to infinite injury priority arguments. Suppose that a node $\sigma \in T^2$ has initial derivative (defined inductively in Step 4) ν along a path Λ^1 through T^1, and principal derivative $\pi \supset \nu$ along $\eta \subset \Lambda^1$. Then we form a primary η-link $[\nu,\pi]$ from ν to π, thereby restraining any node $\xi \in [\nu,\pi)$ from acting and destroying computations declared by π. (Note that if $[\nu,\pi]$ is an η-link, then π is not restrained by $[\nu,\pi]$. However, as we can have $[\nu,\pi) = [\nu,\delta)$ as intervals with $\pi \neq \delta$, we use closed interval notation $[\nu,\pi]$ for η-links to make sure that there is a one-to-one correspondence between intervals which determine links, and the links themselves.) Any such ξ will either be a derivative of a node which is no longer on the approximation to the true path, or a derivative of a node $\rho \subseteq \sigma$. The links prevent derivatives of $\rho \subseteq \sigma$ from acting. Derivatives of such a node ρ which lie beyond the link $[\nu,\pi]$ will be able to act, and we will show that there is no harm in preventing derivatives of ρ restrained by the link from acting. We will allow derivatives of π to act, and so do not restrain π in this link.)

A node $\eta \in T^k$ such that $lh(\eta) > 0$ is said to be *switching* if there is an $r > k$ such that $\lambda^r(\eta^-)|\lambda^r(\eta)$. For the least such r, we say that η is *r-switching*. If $j \in [r,n]$ and η is r-switching, we say that η *switches* $up^j(\eta^-)$.

Fix $\eta \in T^k$. Each η-link will be derived from a primary η-link. We define the *η-links* of T^k by induction on n-k. If k = n, then there are no η-links. Suppose that k < n.

We first determine the *primary η-links*. Suppose that $\xi \subseteq \eta$ and ξ^- is the principal derivative of $\gamma = up(\xi^-)$ along η, but is not the initial derivative of γ along η. Let μ be the initial derivative of γ along η. Then $[\mu,\xi^-]$ is a *primary η-link.*

η-links can also be created by pulling down $\lambda(\eta)$-links. Suppose that $[\rho,\tau]$ is a $\lambda(\eta)$-link on T^{k+1}. Then the initial derivative μ of ρ along η and the principal derivative π of τ along η will exist. $[\mu,\pi]$ is an *η-link derived from* $[\rho,\tau]$.

If $[\rho,\tau]$ is derived from some link $[\zeta,\kappa]$, then every link derived from $[\rho,\tau]$ is *derived from* $[\zeta,\kappa]$. If r > k, then an *η-link on* T^r is just a $\lambda^r(\eta)$-link. We say that ξ *is η-restrained* if there is an η-link $[\mu,\pi]$ such that $\mu \subseteq \xi \subset \pi$. In this case, we say that ξ *is η-restrained by* $[\mu,\pi]$. ξ *is η-free* if ξ is not η-restrained.

Step 3: η-consistency. We decide, in this step, whether a node $\sigma \in T^{k+1}$ is allowed to assign subrequirements at η. This will depend on four conditions. The first condition, (2.7), requires η to predict that σ is on the true path of T^r for all $r \in [k,n]$. The second condition, (2.8), requires that if $\sigma \in T^{k+1}$, once a witness $\xi \subset \eta$ for an existential sentence associated with σ is found, no derivatives of σ can extend ξ. In this case, η has all the information needed to correctly predict the outcome of σ. However, we do not search for such witnesses on T^k if $k \geq \dim(\sigma)$, as we have not yet begun to decompose the sentence assigned to σ in this case. Rather, we will require σ to code the outcome of a unique derivative of σ on T^k, and so impose condition (2.9) requiring that there be a unique such derivative. Condition (2.10) requires that σ be $\lambda(\eta)$-free. (We note that the definition of η-consistency is the same as that in [LL2], but differs from the definition in [LL1].) (2.10) implies (2.7), but it is convenient to require both, before the implication is apparent.

For $\eta \in T^k$, we say that $\sigma \in T^{k+1}$ *is η-consistent* if the following conditions hold:

(2.7) $\quad up^r(\sigma) \subseteq \lambda^r(\eta)$ for all $r \in [k+1,n]$.

(2.8) $\quad$ If $\sigma \subset \lambda(\eta)$, then for all $\nu \subset \eta$, if $up(\nu) = \sigma$ and $\dim(\sigma) > k$, then ν has finite outcome along η.

(2.9) $\quad$ For all $\nu \subset \eta$, if $\dim(\sigma) \leq k$ then $up(\nu) \neq \sigma$.

(2.10) $\quad \sigma$ is $\lambda(\eta)$-free.

Step 4: Assignment of Derivatives. Let $\eta \in T^k$ be given such that requirements have been assigned to all predecessors of η, but not to η. We want to assign a requirement (or subrequirement) to η. The requirement chosen will either be one which does not have any derivatives along η, or one assigned to some η-consistent node of T^k for some k.

Requirements are assigned in *blocks*. (Blocks on T^0 are the counterpart of *stages* in the usual approach to priority constructions; the node which begins a block is the counterpart of a stage, and the remaining nodes within the block play the role of substages of that stage.) If $k = n$ and R_i is the highest priority requirement which has not yet been assigned to a node $\subset \eta$, then we assign the first node of the module for R_i to η, and the second node of the module for R_i, if it exists, to every $\xi \supset \eta$ such that $\xi^- = \eta$ and η has infinite outcome along ξ. The *η-block* of T^2 consists of the nodes of T^2 to which the requirement R_i has just been assigned. (Thus if the basic module for R_i has a single node, then the η-block is $\{\eta\}$.) If $\zeta \supset \eta$, then a *ζ-path through the η-block is completed at δ* if $\delta \subset \zeta$, δ is in the η-block, and there is no γ such that $\delta \subset \gamma \subseteq \zeta$ and γ is in the η-block.

Suppose that $k < n$. A *$\langle\,\rangle$-block* on T^k is *begun* at $\langle\,\rangle$. A *σ-path through the ν-block is completed* at $\delta \in T^k$ if $\nu \subseteq \delta \subset \sigma$, no γ such that $\nu \subseteq \gamma \subset \delta$ completes a σ-path through the ν-block, $up(\delta)$ completes a $\lambda(\sigma)$-path through a block of T^{k+1} and δ is an initial derivative of up(δ). κ is in the *ν-block* if $\nu \subseteq \kappa$ and we have not completed a κ-path through the ν-block at any $\delta \subset \kappa$. If no such ν exists, then we *begin* a *κ-block* at κ, and place κ in the κ-block.

Fix ν such that η is in the ν-block. There are two substeps to consider.

Substep 4.1: If either $\eta\square = \langle\,\rangle$, $\eta\square = \nu$, or η is switching, set $\rho = \langle\,\rangle$. In this case, we start an *η-subblock*. Otherwise, fix $\rho \subseteq \lambda(\eta)$ such that $\rho^- = \mathrm{up}(\eta^-)$. (By induction using (2.7), $\mathrm{up}(\eta^-) \subseteq \lambda(\eta^-)$ and η provides an outcome for a derivative of $\mathrm{up}(\eta^-)$; hence by (2.7), $\mathrm{up}(\eta^-) \subset \lambda(\eta)$ so such a $\rho\square$ must exist.) Search for an η-consistent $\sigma^{k+1} \in T^{k+1}$ of shortest possible length such that $\rho \subseteq \sigma^{k+1}$. (We note that for any $j \geq k$, any $\lambda^j(\eta\square)$-link $[\mu^j,\pi^j]$ satisfies $\pi^j \subset \lambda^j(\eta\square)$, so $\lambda(\eta)$ is $\lambda(\eta)$-free. Furthermore, (2.7) for k will follow from (2.7) for k+1. It thus follows that $\lambda(\eta)$ is η-consistent, so σ^{k+1} must exist.) Let R_i be the requirement assigned to σ^{k+1}. We assign R_i to η, and designate η as a *derivative* of σ^{k+1}. If $k = 0$, or if $k = 1$, $\xi^- = \eta$ and η has finite outcome along ξ, then Step 4 for η is complete and we begin Step 4 for ξ next. Otherwise, we go to Substep 4.2 below for ξ.

Substep 4.2: (The role of this substep is to protect one of the computations $\Phi_e(A_i;x) = \Xi(\emptyset;x)$, when the computation $\Phi_e(A_j;x)$ for $j = 1-i$ has been destroyed. The danger is that there is no node along ξ to which x is assigned and which checks if $\Phi_e(A_0;x) = \Phi_e(A_1;x)$; so we specify such a node. This prevents a node of T^2 which predicts that $\Phi_e(A_0;x) = \Phi_e(A_1;x)$ and which might destroy the computation $\Phi_e(A_i;x) = \Xi(\emptyset;x)$ from being consistent until we have recovered a new computation yielding $\Phi_e(A_j;x) = \Xi(\emptyset;x)$. This substep has no counterpart in [LL2].) We iterate this substep, beginning at $\xi \in T^1$ and assigning requirements to $\gamma \supseteq \xi$, until either we fail to find ρ below, or we reach $\gamma \supset \xi$ such that γ^- has infinite outcome along γ. When we decide, at γ, that this substep will no longer be followed, we return to the beginning of Step 4 for γ. Suppose that either $\gamma = \xi$ or that γ^- has finite outcome along γ. We search for the shortest ξ-consistent $\rho \in T^2$ of type 1 such that ρ has a derivative along η and $|\{\mu \subset \gamma\colon \mathrm{up}(\mu) = \rho\}| < \mathrm{wt}(\xi)$. If such a ρ exists, then the requirement assigned to μ is the requirement which has previously been assigned to this ρ.

Suppose that we have just assigned a requirement R to $\eta \in T^k$. Let η be a derivative of $\sigma^{k+1} \in T^{k+1}$. Then we set $\mathrm{up}(\eta) = \sigma^{k+1}$ and assign a type and dimension to η in the same way as these were assigned to R. The derivative operation can be iterated; thus for every ζ such that σ^{k+1} is a derivative of ζ, we call η a *derivative* of ζ. η is also a derivative of η. If $r > k$, $\zeta \in T^r$, and η is a derivative of ζ then we write $up^r(\eta) = \zeta$. If there is no $\xi \subset \eta$ such that $up(\xi) = \sigma^{k+1}$, then for all $\nu \supseteq \eta$, we call η the *initial derivative of* σ^{k+1} *along* ν, and if, in addition, σ^{k+1} is the initial derivative of ζ along σ^{k+1}, then η is the *initial derivative of* ζ *along any* $\nu \supseteq \eta$. ζ is an *antiderivative* of ξ if ξ is a derivative of ζ.

If Λ^k is a path through T^k, then we let $\lambda(\Lambda^k) = \lim_s\{\lambda(\Lambda^k|s)\}$, and define $\Lambda^{k+1} = \lambda(\Lambda^k)$. (We will show in Lemma 3.2 that if $lh(\Lambda^k) = \infty$, then $lh(\Lambda^{k+1}) = \infty$.) For $\Lambda^k \in [T^k]$, $[\mu^k,\pi^k]$ is a *primary* Λ^k*-link* if there is a ξ^k ⊂ Λ^k such that $[\mu^k,\pi^k]$ is a primary ξ^k-link. $[\mu^k,\pi^k]$ is a Λ^k*-link* if it is derived from a primary Λ^j-link for some $j \geq k$. The notions of ξ *is* Λ^k*-restrained,* and ξ *is* Λ^k*-free* are now defined as in Step 2, with Λ^k in place of η. ■

Suppose that $k < n$, $\sigma \subset \tau \in T^k$, and $\lambda(\sigma) \neq \lambda(\tau)$. By (2.5), $out(\lambda(\sigma)) \subseteq \sigma$ and $out(\lambda(\tau)) \subseteq \tau$, so by (2.4) and (2.5) and as $\sigma \subset \tau$, $out(\lambda(\sigma)) \subseteq \sigma \subset out(\lambda(\tau)) \subseteq \tau$. By (2.4) and (2.5), for all $\kappa \subseteq \lambda(\sigma)$, $out(\kappa) \subseteq out(\lambda(\sigma)) \subset out(\lambda(\tau))$. It now follows from (2.3) that:

(2.11) For all $k < n$ and $\sigma,\tau \in T^k$, if $\sigma \subset \tau$ then $wt(\lambda(\sigma)) \leq wt(\lambda(\tau))$.

Our first lemma will tell us that if $\Lambda^0 \in [T^0]$ is infinite, then Λ^1 and Λ^2 are also infinite. It also provides information on the value of λ on initial and principal derivatives.

Lemma 2.1: Fix $k \in [0,n)$ and a path $\Lambda^k \in [T^k]$. Then:

(i) If $\sigma \subset \lambda(\Lambda^k)$, then σ has an initial derivative ν along Λ^k and $\lambda(\nu) = \sigma$.

(ii) If $\sigma \subset \lambda(\Lambda^k)$, then there is a $\pi \subseteq \Lambda^k$ such that π^- is the principal derivative of σ along Λ^k, $\lambda(\pi)^- = \sigma$, and for all $\eta \subseteq \Lambda^k$, $\lambda(\pi) \subseteq \lambda(\eta)$ iff $\pi \subseteq \eta$.

(iii) If $\sigma \subset \tau \subset \lambda(\rho) \subseteq \lambda(\Lambda^k)$, μ (ν, resp.) is the initial derivative of σ (τ, resp.) along ρ, and π (δ, resp.) is the principal derivative of σ (τ, resp.) along ρ, then $\mu \subseteq \pi \subset \nu \subseteq \delta$.

(iv) If $lh(\Lambda^k) = \infty$, then for any δ-block such that $\delta \subset \Lambda^k$, there is a $\xi \subset \Lambda^k$ such that ξ completes a path through the δ-block.

(v) If $lh(\Lambda^k) = \infty$, then $\Lambda^{k+1} = \lambda(\Lambda^k) = \lim\{\lambda(\eta): \eta \subset \Lambda^k\}$ exists and $lh(\Lambda^{k+1}) = \infty$.

Proof: We proceed by induction on n-k.

(i): By (2.4) and as $\sigma \subset \lambda(\Lambda^k)$, σ must have a derivative along Λ^k. Hence if ν is the shortest derivative of σ along Λ^k, then ν is the initial derivative of σ along Λ^k. By (2.7), $\lambda(\nu) \supseteq \sigma$. As no derivative of σ has an outcome along ν, it follows from (2.4) that, $\lambda(\nu) = \sigma$.

(ii): If $\dim(\sigma) \leq k$, then by (2.9), the initial derivative ν of σ along Λ^k is the principal derivative of σ along Λ^k. (ii) follows in this case from (i), and as by (2.4) and (2.7), no $\tau \supset \sigma$ can have a derivative $\mu \subset \nu$.

Suppose that $\dim(\sigma) > k$. By (i), let ν be the initial derivative of σ along Λ^k. If there is no $\pi \subset \Lambda^k$ such that $up(\pi^-) = \sigma$ and π^- has infinite outcome along π, then it follows as in the case for $\dim(\sigma) \leq k$ that ν is the principal derivative of σ along Λ^k. Otherwise, fix the shortest such π. We note that π^- is the principal derivative of σ along Λ^k. By (2.4), induction, (2.7) and (2.6), $\lambda(\pi) = \sigma\hat{\ }\langle\pi\rangle \subseteq \Lambda^{k+1}$, and if $\eta \subset \Lambda^k$ then $\lambda(\eta) \supseteq \sigma\hat{\ }\langle\pi\rangle$ iff $\eta \supseteq \pi$.

(iii): It follows easily from (2.4) that $\mu \subseteq \pi$ and $\nu \subseteq \delta$. By (i), $\lambda(\nu) = \tau \supset \sigma = \lambda(\mu)$. By (2.4) and (2.5), if $\gamma \subseteq \lambda(\rho)$ and $\gamma^- = \sigma$, then $(out(\gamma))^- = \pi$ and $out(\gamma) \subseteq \nu$. Hence $\pi \subset \nu$.

(iv),(v): It follows easily from (2.7) that $\Lambda^{k+1} = \lambda(\Lambda^k) = \lim\{\lambda(\eta)\colon \eta \subset \Lambda^k\}$ exists. First suppose that $lh(\Lambda^{k+1}) = \infty$. By (iv) inductively, there are infinitely many blocks along Λ^{k+1} (note that this is immediate for $k = n$), so there are infinitely many $\tau \subset \Lambda^{k+1}$ such that τ completes a Λ^{k+1}-path through a block. By (i), each such τ has an initial derivative along Λ^k. Hence by Step 4, there are infinitely many $\xi \subset \Lambda^k$ which complete Λ^k-paths through blocks, and (iv) and (v) hold in this case.

Now suppose that $lh(\Lambda^{k+1}) < \infty$ in order to obtain a contradiction. Then by (2.4), there is a $\delta \subset \Lambda^k$ such that for all ν satisfying $\delta \subseteq \nu \subset \Lambda^k$, $\lambda(\nu) = \Lambda^{k+1}$. If $\delta \subseteq \nu \subset \Lambda^k$ and ν completes a path through a block, then ν must be an initial derivative of some node $\subseteq \Lambda^{k+1}$. As this is possible only finitely often and $lh(\Lambda^k) = \infty$, we can assume without loss of generality that there is no ν such that $\delta \subseteq \nu \subset \Lambda^k$ and ν completes a path through any block. By (2.4), (2.6) and Step 4, if $\delta \subseteq \nu \subset \gamma \subset \Lambda^k$ then ν is nonswitching, so $up(\nu) \subseteq up(\gamma) \subset \Lambda^{k+1}$. Furthermore if we fix ξ as in Substep 4.2, then for any γ in the same block as δ, $|\{\nu\colon \exists\gamma(\delta \subseteq \nu \subset \gamma \subset \Lambda^k \,\&\, up(\nu) = up(\gamma))\}| < wt(\xi)$. But this is impossible if $lh(\Lambda^{k+1}) < \infty$ and $lh(\Lambda^k) = \infty$. ■

From now on, whenever we write $\Lambda^k \in [T^k]$, we assume that there is a $\Lambda^0 \in [T^0]$ such that $\Lambda^k = \lambda^k(\Lambda^0)$. Similarly, if we write $\eta \in T^k$, we assume that $\eta \subset \Lambda^k$ for some $\Lambda^k \in [T^k]$. If this is not the case, then η and Λ^k are irrelevant to our construction.

Our next lemma provides sufficiently many free derivatives of nodes on the true path so that the construction can act to satisfy all requirements. We first note an important fact, whose proof we leave to the reader.

(2.12) Fix $k \le n$, $\Lambda^k \in [T^k]$, and $\mu \subset \nu \subset \eta$ such that $up^j(\eta) \subseteq \lambda^j(\eta)$ for all j such that $k \le j \le n$. Then $[\mu,\nu]$ is a Λ^k-link iff $[\mu,\nu]$ is an η-link.

Lemma 2.2: Fix $k < 2$ and $\Lambda^k \in [T^k]$ such that $lh(\Lambda^k) = \infty$, and for all $r \in [k,2]$, let $\Lambda^r = \lambda^r(\Lambda^k)$. Suppose that $\sigma \subset \Lambda^{k+1}$ is Λ^{k+1}-free. Then:

(i) If $\pi \subset \Lambda^k$ is the principal derivative of σ along Λ^k, then π is Λ^k-free.

(ii) If σ has infinite outcome along Λ^{k+1} and $\dim(\sigma) > k$, then there are infinitely many Λ^k-free derivatives of σ.

Proof: (i): First suppose that $[\mu,\nu]$ is a primary Λ^k-link which restrains π in order to obtain a contradiction. As μ□ is not the principal derivative of $up(\mu)$ along Λ^k, $\mu \neq \pi$. So $\mu \subset \pi \subset \nu$, $up(\mu) = up(\nu)$, and ν has infinite outcome along Λ^k. By (2.7), $up(\mu) \subseteq \lambda(\mu),\lambda(\nu)$ and $up(\pi) \subseteq \lambda(\pi)$. As $\mu \subset \pi \subset \nu$, it follows from (2.6) that $up(\mu) \subseteq \lambda(\pi)$. Hence by Lemma 2.1(iii), $up(\mu) \subset up(\pi) = \sigma$. Now if $\xi \subset \Lambda^k$ and $\xi^- = \nu$, then ξ switches $up(\mu)$, so by (2.6) $\sigma \not\subset \Lambda^k$ contrary to hypothesis.

Suppose that $[\mu,\nu]$ restrains π and is derived from a Λ^j-link for some $j > k$ in order to obtain a contradiction. Since there are no links on T^2, all links on T^1 are primary links so $k \neq 1$. Thus $k = 0$ and $[up(\mu),up(\nu)]$ is a primary Λ^1-link. By (2.6), $up(\mu) = up(\nu)$ and $up(\pi)$ are comparable. So by Lemma 2.1(iii), $up(\mu) \subset up(\pi) \subset up(\nu)$, a contradiction.

(ii): We note by (i) that if $\zeta \subset \Lambda^n$, then for all j such that $k \le j \le n$, the principal derivative ζ^j of ζ along Λ^j is Λ^j-free. As $lh(\Lambda^k) = \infty$, it follows inductively from Lemma 2.1(v) that $lh(\Lambda^j) = \infty$ for all j such that $k \le j \le n$. Hence there are infinitely many $\zeta \subset \Lambda^n$ such that ζ^j extends $up^j(\sigma)$ for all j such that $k \le j \le n$. Fix such a ζ. If suffices to show that σ has a free derivative along Λ^k which extends ζ^k.

By (2.4), if we fix $\gamma^k \subset \Lambda^k$ such that $(\gamma^k)^- = \zeta^k$, then γ^k is Λ^k-true. Hence by (2.12), for all j such that $k \le j \le n$ and $\delta^j \subseteq \lambda^j(\gamma^k)$, δ^j is Λ^j-free iff δ^j is $\lambda^j(\gamma^k)$-free. In particular, σ is $\lambda(\gamma^k)$-free. It is routine to verify that every Λ^k-free node is Λ^k-true. (For by contradiction, if we were to fix the least $j > k$ such that the Λ^k-free node τ satisfies $up^j(\tau) \not\subset \Lambda^j$, then $up^{j-1}(\tau)$ would be restrained by a primary Λ^j-link, and the Λ^k-link derived from this Λ^j-link would restrain τ.) It thus follows by hypothesis that σ is γ^k-consistent. Furthermore, either γ^k is switching, or $(\gamma^k)^-$ is the initial derivative of ζ along Λ^k. In either case, we set $\rho = \langle\,\rangle$ at the beginning of Subtep 4.1, just prior to assigning a requirement to γ^k. As $\sigma \subset \lambda(\gamma^k) \subset \Lambda^{k+1}$, it follows from (2.4) and (2.6) that we will take nonswitching extensions in Step 4, beginning at γ^k, and reach a node β^k at which σ is the shortest node eligible to determine a derivative along Λ^k. We note that no new links are formed when nonswitching extensions are taken, and that, as $\sigma \subset \lambda(\gamma^k)$, it will be the case that $\lambda(\beta^k) = \lambda(\gamma^k)$, and so, that σ is β^k-free and β^k-consistent. By Step 4, we define $up(\beta^k) = \sigma$, and β^k will be Λ^k-true and Λ^k-free. Hence by (2.12), β^k will be a Λ^k-free derivative of σ. ■

We can now show that the construction will have to deal with all requirements.

Lemma 2.3: Fix $\Lambda^0 \in T^0$, and let $\Lambda^2 = \lambda^2(\Lambda^0)$. Then the requirement R_i is assigned to some node $\sigma \subset \Lambda^2$.

Proof: Each block along Λ^2 consists of a single module. The lemma now follows from Lemma 2.1(ii), as there are infinitely many blocks along Λ^2, and Substep 4.3. ■

We note that, with the exception of Lemma 2.1(i), the lemmas of this section can be proved under very general assumptions, and that this is done in [LL2]. Hence much of what is proved in this section can be obtained from a general framework, and need not be repeated for each construction.

3. The Construction and Proof for a Minimal Pair. Our construction will define an infinite recursive $\Lambda^0 \in [T^0]$, and for $k \leq 1$, we will set $\Lambda^{k+1} = \lambda(\Lambda^k)$. For $i \leq 1$, we construct partial recursive functionals which are total on oracle A_i such that for all e, there is an x such that $\Phi_e(\emptyset;x) \neq \Delta_i(A_i;x)$. And for all $\sigma \in T^2$ such that σ has type (1,1), we construct a partial recursive functional Ξ_σ such that for the e mentioned in the requirement assigned to σ, if $\Phi_e(A_0) = \Phi_e(A_1)$ is total and $\sigma \subset \Lambda^2$, then $\Xi_\sigma(\emptyset)$ is total and for all x, $\Phi_e(A_0;x) = \Xi_\sigma(\emptyset;x)$.

Definition 3.1: Given $\eta \subset \Lambda^0$, the action taken by the construction at η will depend on the truth or falsity of a sentence M_η associated with η. If η has type 0 and is working on the requirement $\Phi_e(\emptyset) \neq \Delta_i(A_i)$, then M_η will be the quantifier-free sentence $\Phi_{e,wt(\eta)}(\emptyset;wt(up(\eta)))\downarrow = 0$ with use $\leq wt(\eta)+1$. Suppose that η has type (1,0) or (1,1), and $|\tau \subset up(\eta)\colon up(\tau) = up^2(\eta)| = x$. If η has type (1,0), then M_η is the quantifier-free sentence $\Phi_{e,wt(\eta)}(A_0^{wt(\eta)};x)\downarrow$ and $\Phi_{e,wt(\eta)}(A_1^{wt(\eta)};x)\downarrow$ with uses $\leq wt(\eta)+1$. And if η has type (1,1), then M_η is the quantifier-free sentence $\Phi_{e,wt(\eta)}(A_0^{wt(\eta)};x)\downarrow = \Phi_{e,wt(\eta)}(A_1^{wt(\eta)};x)\downarrow$ with uses $\leq wt(\eta)+1$.

The Construction: We begin by specifying that $\langle\ \rangle \subset \Lambda^0$. No axioms $\Delta_{i,0}(A_i^0;x)$ or $\Xi_{\sigma,0}(\emptyset;x)$ are declared, and $A_i^0 = \emptyset$ for all $i \leq 1$. (We assume, without loss of generality, that $wt(\langle\ \rangle) = 0$.)

Suppose that we have specified that $\eta \subset \Lambda^0$, and are ready to determine the immediate successor of η along Λ^0. We specify that $\xi = \eta\hat{}\langle 0\rangle \subset \Lambda^0$ if M_η is false, and that $\xi = \eta\hat{}\langle\infty\rangle \subset \Lambda^0$ if M_η is true, and proceed by cases, depending on the type of η. $A_j^{wt(\xi)} = A_j^{wt(\eta)}$ for $j \in \{0,1\}$ and no new axioms are declared, unless we specify otherwise below. Also, the axioms declared below are declared only if they are consistent with previously declared axioms. (We will show later that this is always the case.)

Case 1: η has type 0. If η is working on the requirement $\Phi_e(\emptyset) \neq \Delta_i(A_i)$ and M_η is false, define $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};wt(up(\eta))) = 0$ with use $wt(up(\eta))+1$. If M_η is true, set $A_i^{wt(\xi)} = A_i^{wt(\eta)} \cup \{wt(up(\eta))\}$ if η is not the initial derivative of $up(\eta)$ along ξ□, and set $A_i^{wt(\xi)} = A_i^{wt(\eta)}$ if η is the initial derivative of $up(\eta)$ along ξ□. In both cases, define $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};wt(up(\eta))) = 1$ with use $wt(up(\eta))+1$.

Case 2: η has type (1,0). No action is taken.

Case 3: η has type (1,1). If M_η is false, then no action is taken. If M_η is true, then $\Phi_{e,wt(\eta)}(A_0^{wt(\eta)};x)\downarrow = \Phi_{e,wt(\eta)}(A_1^{wt(\eta)};x)\downarrow = y$ for the x associated with M_η and some y. We declare the axiom $\Xi_{up^2(\eta),wt(\xi)}(\emptyset;x) = y$ with use 1.

For all x and i such that no axiom $\Delta_i(A_i;x) = m$ has been declared above for any m, we set $\Delta_i(A_i;x) = 0$. (This will cause $\Delta_i(A_i)$ to be total. We will show in Lemma 3.1 that

$\Delta_i(A_i)$ is recursive in A_i.) And for all x and $\sigma \in T^2$ such that no axiom $\Xi_\sigma(\emptyset;x) = m$ has been declared above for any m, we set $\Xi_\sigma(\emptyset;x) = 0$. ■

We now show that the functionals defined above are partial recursive. As new axioms are declared only when they are consistent with previously declared axioms, these functionals are well-defined.

Lemma 3.1 (Recursiveness Lemma): (i) For all $i \leq 1$, $\Delta_i(A_i)$ is a well-defined total function which is recursive in A_i.

(ii) For all $\tau \subset \Lambda^2$ of type (1,1) such that τ has infinite outcome along Λ^2, $\Xi_\tau(\emptyset)$ is a total recursive function.

Proof: (i): Fix $i \leq 1$. By the last paragraph of the construction, $\Delta_i(A_i)$ is total. (When we write $\Delta_{i,s}(A_i^s;x)\uparrow$ below in this proof, our intent is to have this mean that no axiom defined in Step 1 forces a convergent computation; we ignore the axioms specified at the end of the construction.) Let $S = \{x: \Delta_i(A_i;x)$ is not defined by Case 1 of the construction$\}$. Fix x. We claim that $x \in S$ iff there is an $\eta \subset \Lambda^0$ such that $\Delta_{i,wt(\eta)}(A_i^{wt(\eta)};x)\uparrow$, $wt(\lambda(\eta)) > x$ and there do not exist e and $\xi \subseteq \eta$ such that $up(\xi) \subset \lambda(\eta)$, $wt(up(\xi)) = x$, ξ has type 0, and the requirement $\Phi_e(\emptyset) \neq \Delta_i(A_i)$ is assigned to ξ. Thus S is r.e., so we can define a partial recursive functional Δ such that $\Delta(A_i) = \Delta_i(A_i)$.

To see the claim, first suppose that $x \notin S$. Then $x = wt(\rho)$ for some $\rho \in T^1$. Since $x \notin S$, an axiom $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};x) = m$ must be defined by Case 1 when $\xi \subset \Lambda^0$ and $\mu = \xi^-$ is the initial derivative of ρ along Λ^0, and this axiom will have use $wt(\rho)+1$. By (2.2), $x = wt(\rho) = wt(\lambda(out(\rho))) > wt(out(\rho))$. A new axiom can be defined by Case 1 at most once more, at δ where $\pi = \delta^-$ is the principal derivative of ρ along Λ^0. If this happens, it follows from (2.6) that $\rho \subseteq \lambda(\nu)$ for all ν such that $out(\rho) \subseteq \nu \subseteq \pi$. Now only elements of the form $wt(\lambda(\nu))$ can be placed in sets at such ν, and by (2.1), $wt(\lambda(\nu)) > wt(\rho)$, so the axiom defined at ξ is not injured before δ. If a new axiom is defined at δ, then we set $\Delta_{i,wt(\delta)}(A_i^{wt(\delta)};x) = 1$ in Step 1, and this axiom will have use $wt(\rho)+1$. If this axiom (or the axiom defined at ξ if δ does not exist) is destroyed at γ, then no axiom for x is ever redefined in Case 1, so $x \in S$, contrary to assumption. So no η as above can exist.

Now suppose that $x \in S$. By (2.1)-(2.3), we can recursively determine whether $x = wt(\rho)$ for some $\rho \in T^1$. If not, then any $\eta \subset \Lambda^0$ such that $wt(\lambda(\eta)) > x$ will have the desired properties, and by Lemma 2.1(v), such an η must exist. Suppose that ρ exists. We can recursively determine whether ρ has a derivative along Λ^0. For by (2.1) and Lemma

2.1, we can find $\gamma \subset \Lambda^0$ such that $\mathrm{wt}(\lambda(\gamma)) > \mathrm{wt}(\rho\square)$, and by (2.11) and Lemma 2.1, ρ has a derivative along Λ^0 iff ρ has a derivative along γ. If no such γ exists, then any $\eta \subset \Lambda^0$ such that $\mathrm{wt}(\lambda(\eta)) > x$ will have the desired properties, and by Lemma 2.1(v), such an η must exist. Suppose that γ exists. Let μ be the initial derivative of ρ along Λ^0.

An axiom $\Delta_i(A_i;x) = m$ defined during Case 1 must be defined at μ, and this axiom will have use $\mathrm{wt}(\rho)+1$. Furthermore, if $\Delta_{i,\mathrm{wt}(\nu)}(A_i^{\mathrm{wt}(\nu)};x)\uparrow$ at any ν such that $\mu \subseteq \nu \subset \Lambda^0$, then for the least such ν, $\mathrm{wt}(\mathrm{up}(\nu^-)) \le \mathrm{wt}(\rho)$. By (2.1), (2.11), and as $\mathrm{up}(\nu^-) \ne \rho$ else $\Delta_{i,\mathrm{wt}(\nu)}(A_i^{\mathrm{wt}(\nu)};x)\downarrow$ by Step 1, it follows that $\mathrm{up}(\nu^-) \subset \rho$. Hence $\rho \not\subset \lambda(\nu)$, so by (2.3) and Case 1, $\Delta_{i,\mathrm{wt}(\delta)}(A_i^{\mathrm{wt}(\delta)};x)\uparrow$ for all δ such that $\nu \subseteq \delta \subset \Lambda^0$. As the use of any axiom declared for x is bounded by $\mathrm{wt}(\rho)+1$ and $\Delta_i(A_i;x)$ is not defined in Case 1, an η with the desired properties must exist.

(ii): As τ has infinite outcome along Λ^2, it follows from Lemma 2.2 that τ has infinitely many derivatives along Λ^1. If ξ is such a derivative, $|\{\nu \subset \xi: \mathrm{up}(\nu) = \tau\} = x$, and π is the principal derivative of ξ along Λ^0, then Case 3 defines an axiom $\Xi_{\tau,\mathrm{wt}(\pi)}(\emptyset,x) = m$ for some m. ■

We now show that all requirements are satisfied. We begin with the requirements of type 0.

Lemma 3.2: For all e and all $i \le 1$, $\Phi_e(\emptyset) \ne \Delta_i(A_i)$.

Proof: Fix j such that R_j is the requirement $\Phi_e(\emptyset) \ne \Delta_i(A_i)$. By Lemma 2.3, R_j is assigned to a unique $\sigma \subset \Lambda^2$. By (2.9) and Lemma 2.2, there is a unique $\nu \subset \Lambda^1$ such that $\mathrm{up}(\nu) = \sigma$, and ν is Λ^1-free.

First suppose that ν has infinite outcome along Λ^1. By Lemma 2.1(v) and Lemma 2.2(ii), ν has infinitely many derivatives along Λ^0, each having finite outcome along Λ^0. By the construction, $\Phi_{e,\mathrm{wt}(\eta)}(\emptyset,\mathrm{wt}(\nu))\uparrow$ or $\Phi_{e,\mathrm{wt}(\eta)}(\emptyset;\mathrm{wt}(\nu))\downarrow \ne 0$ for each such derivative η. Fix $\delta \subset \Lambda^0$ such that $\mu = \delta^-$ is the initial derivative of ν along Λ^0. By Case 1, an axiom $\Delta_{i,\mathrm{wt}(\delta)}(A_i^{\mathrm{wt}(\delta)};\mathrm{wt}(\nu)) = 0$ will be defined in Case 1 with use $\mathrm{wt}(\nu)+1$. As $\nu \subset \Lambda\square^1$ and μ is the principal derivative of ν along Λ^0, it follows from (2.6) that only elements $\mathrm{wt}(\sigma)$ for $\sigma \supset \nu$ will be placed into sets at any $\gamma \supset \mu$ such that $\gamma \subset \Lambda^0$. By (2.1), for any such σ, $\mathrm{wt}(\sigma) > \mathrm{wt}(\nu)$. Hence $\Phi_e(\emptyset;\mathrm{wt}(\nu)) \ne 0 = \Delta_i(A_i;\mathrm{wt}(\nu))$.

On the other hand, suppose that ν has finite outcome ξ along Λ^1. Let $\eta = \xi^-$. Then $\mathrm{up}(\eta) = \nu$ and η is the principal derivative of ν along Λ^0. As in the preceding paragraph,

$\Phi_{e,wt(\eta)}(\emptyset;wt(\nu))\downarrow = 0$ with use $wt(\nu)+1$, and we place $wt(\nu) \in A_i^{wt(\xi)}-A_i^{wt(\eta)}$. As any axiom $\Delta_{i,wt(\eta)}(A_i^{wt(\eta)};wt(\nu)) = m$ has use $wt(\nu)+1$, we consistently redefine $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};wt(\nu)) = 1$ with use $u = wt(\nu)+1$. As $\nu \subset \Lambda^1$, it follows from (2.6) that there is no $\delta \supset \eta$ such that $\delta \subset \Lambda^0$, $up(\delta) \subset \nu$, and δ has infinite outcome along Λ^0. By (2.1) and the construction, $A_i^{wt(\eta)}|u = A_i|u$, so $\Delta_i(A_i;wt(\nu)) = 1$. Hence the lemma follows. ■

Lemma 3.3: Let R_j be the requirement which asserts that if $\Phi_e(A_0) = \Phi_e(A_1)$ is total, then there is a recursive function Ξ such that $\Phi_e(A_0) = \Xi(\emptyset)$. Then R_j is satisfied.

Proof: By Lemma 2.3, R_j is assigned to a unique $\sigma \subset \Lambda^2$ of type (1,0). By Lemma 2.2, σ has a principal derivative $\sigma^1 \subset \Lambda^1$, and σ^1 is free.

First suppose that σ^1 has infinite outcome along Λ^1. Let σ^1 be associated with argument x (so there are x derivatives of σ which are $\subset \sigma^1$). By Lemma 2.1(v) and Lemma 2.2, σ^1 has infinitely many derivatives along Λ^0, and all have finite outcome along Λ^0. By the construction, there are infinitely many s such that either $\Phi_{e,s}(A_0^s,x)\uparrow$ or $\Phi_{e,s}(A_1^s,x)\uparrow$. Thus either $\Phi_e(A_0;x)\uparrow$ or $\Phi_e(A_1;x)\uparrow$, so R_j is satisfied.

Suppose that σ^1 has finite outcome along Λ^1, and fix $\delta^1 \subset \Lambda^1$ such that $(\delta^1)^- = \sigma^1$. Then $\tau^2 = \sigma\hat{\ }\langle\delta^1\rangle$ is the immediate successor of σ along Λ^2, τ^2 and σ are part of the same module, and τ^2 has type (1,1). There are two cases to consider, depending on the outcomes we see for τ^2 during the course of the construction.

Case 1: There is a $\xi^0 \subset \Lambda^0$ such that if $\xi^1 = \lambda(\xi^0)$ and $\tau^1 = (\xi^1)^-$, then $up(\tau^1) = \tau^2$ and τ^1 has infinite outcome along ξ^1. Let τ^1 be associated with argument x (so there are x derivatives of τ^2 which are $\subset \tau^1$). This case corresponds to finding a disagreement at x, since$(\tau^1)^-$ will also be assigned the argument x, will be part of the same module as τ^1, and since the case assumption implies that all derivatives of σ have finite outcome along τ^1, $\Phi_{e,s}(A_0^s,x)\downarrow$ and $\Phi_{e,s}(A_1^s,x)\downarrow$ for $s = wt((out(\tau^1)^-)$. We first show that $\xi^1 \subset \Lambda^1$. For suppose not in order to obtain a contradiction, and fix the shortest ν^0 such that $\xi^0 \subset \nu^0 \subset \Lambda^0$, $\mu^1 = up((\nu^0)^-) \subset \xi^1$, and ν^0 switches μ^1. Let $\tilde{\tau}^1$ be the initial derivative of τ^2 along ξ^1. We note that if $\tilde{\tau}^1 \neq \tau^1$, then $[\tilde{\tau}^1,\tau^1]$ is a primary ν^0-link. By (2.10), it must be the case that either $\mu^1 = \tau^1$ or $\mu^1 \subset \tilde{\tau}^1$. We compare the locations of $\mu^2 = up(\mu^1)$ and τ^2 on T^2.

By Lemma 2.1(iii) and (2.7), $\mu^2 \not\supset \tau^2$. Suppose that $\mu^2 \subset \tau^2$. As ν^0 switches μ^1, it follows from the construction that μ^1 has infinite outcome along ξ^1. Thus if $\nu^1 \subseteq \xi^1$ and $(\nu^1)^- = \mu^1$, then $\mu^2\hat{\ }\langle\nu^1\rangle \subseteq \tau^2$. As $\nu^1 \not\subseteq \lambda(\nu^0)$, it follows from (2.6) that $\nu^1 \not\subseteq \Lambda^1$. But then by (2.4), $\tau^2 \not\subset \Lambda^2$, a contradiction.

Suppose, next, that $\mu^2 | \tau^2$. Let $\rho^2 = \mu^2 \wedge \tau^2$, and let ρ^1 be the principal derivative of ρ^2 along τ^1. By Lemma 2.1(iii), $\rho^1 \subset \tau^1$. Hence by the minimality of the choice of ν^0, ρ^1 is the principal derivative of ρ^2 along $\lambda((\nu^0)^-)$. But then $\mu^2 \not\subseteq \lambda^2((\nu^0)^-)$, contradicting (2.7).

It remains to consider the case where $\mu^2 = \tau^2$. By (2.8), ν^0 must switch τ^1, so $\tau^1 = \mu^1$. Let η be the initial derivative of τ^1 along Λ^0, fix $\widetilde{\eta} \subset \Lambda^0$ such that $(\widetilde{\eta})^- = \eta$, and note that, by Lemma 2.1(iii) and (2.4), $\lambda(\widetilde{\eta}) = \xi^1$. By the construction, $\Phi_{e,\mathrm{wt}(\eta)}(A_0^{\mathrm{wt}(\eta)},x)\downarrow \neq \Phi_{e,\mathrm{wt}(\eta)}(A_1^{\mathrm{wt}(\eta)},x)\downarrow$, and the uses of these computations are $\leq \mathrm{wt}(\eta)+1$. By (2.6) and the construction, all elements placed into A_0 or A_1 at nodes γ such that $\eta \subset \gamma \subset \nu^0$ are of the form $\mathrm{wt}(\mathrm{up}(\gamma^-))$ with $\mathrm{up}(\gamma^-) \supseteq \xi^1$. Furthermore, we have shown that $\lambda(\eta) = \tau^1 \subset \xi^1 = \lambda(\widetilde{\eta})$. Hence by (2.1) and (2.2),

$$\mathrm{wt}(\eta) < \mathrm{wt}(\widetilde{\eta}) < \mathrm{wt}(\lambda(\widetilde{\eta})) = \mathrm{wt}(\xi^1) \leq \mathrm{wt}(\mathrm{up}(\gamma^-)).$$

Hence if $\beta = (\nu^0)^-$, then for $i \leq 1$, $\Phi_{e,\mathrm{wt}(\beta)}(A_i^{\mathrm{wt}(\beta)},x)\downarrow = \Phi_{e,\mathrm{wt}(\eta)}(A_i^{\mathrm{wt}(\eta)},x)$ with use $\leq \mathrm{wt}(\eta)+1$. By the construction, ν^0 is nonswitching, contrary to assumption.

We conclude that $\tau^2 \wedge \langle \xi^1 \rangle \subset \Lambda^2$. It follows as in the preceding paragraph that $\Phi_e(A_0;x) = \Phi_{e,\mathrm{wt}(\eta)}(A_0^{\mathrm{wt}(\eta)},x) \neq \Phi_{e,\mathrm{wt}(\eta)}(A_1^{\mathrm{wt}(\eta)},x) = \Phi_e(A_1;x)$, so R_j is satisfied.

Case 2: Otherwise. (In this case, a disagreement is never found.) Then for all $\xi^0 \subset \Lambda^0$ such that $\mathrm{up}^2((\xi^0)^-) = \tau^2$, $\mathrm{up}((\xi^0)^-)$ has finite outcome along $\lambda(\xi^0)$. Fix x, and fix $\tau^1 \subset \Lambda^1$ such that $\mathrm{up}(\tau^1) = \tau^2$ and the argument x is assigned to τ^1. Let η be the initial derivative of τ^1 along Λ^0, and fix $\gamma \subset \Lambda^0$ such that $\gamma^- = \eta$. If $\Xi_{\sigma,\mathrm{wt}(\eta)}(x)\uparrow$, then we define $\Xi_{\sigma,\mathrm{wt}(\gamma)}(x) = \Phi_{e,\mathrm{wt}(\eta)}(A_0^{\mathrm{wt}(\eta)},x) = \Phi_{e,\mathrm{wt}(\eta)}(A_1^{\mathrm{wt}(\eta)},x)$, and note that, as $\tau^1 \subset \Lambda^1$, all elements entering A_0 or A_1 at any γ such that $\eta \subseteq \gamma \subset \Lambda^0$ are of the form $\mathrm{wt}(\kappa\Box)$ for some $\kappa \supset \tau^1$. As in the preceding case, we see that for all such τ^1, $\mathrm{wt}(\tau^1) > \mathrm{wt}(\eta)$. As the use of the computations $\Phi_{e,\mathrm{wt}(\eta)}(A_i^{\mathrm{wt}(\eta)},x)$ is $\leq \mathrm{wt}(\eta)+1$ for $i \leq 1$, it follows that

$$\Phi_e(A_0,x) = \Phi_{e,\mathrm{wt}(\eta)}(A_0^{\mathrm{wt}(\eta)},x) = \Phi_{e,\mathrm{wt}(\eta)}(A_1^{\mathrm{wt}(\eta)},x) = \Phi_e(A_1,x).$$

Hence R_j is satisfied in this case.

Suppose that $\Xi_{\sigma,\mathrm{wt}(\eta)}(x)\downarrow$. Fix the shortest $\xi \subseteq \eta$ such that $\Xi_{\sigma,\mathrm{wt}(\xi)}(x)\downarrow$. We show that for all μ such that $\xi \subset \mu \subseteq \eta$, there is an $i \leq 1$ such that

(3.1) $\Phi_{e,wt(\mu)}(A_i^{wt(\mu)},x)\downarrow = \Phi_{e,wt(\mu^-)}(A_i^{wt(\mu^-)},x)\downarrow = \Xi_\sigma(x)$, and
$A_i^{wt(\mu)}\restriction wt(\mu^-)+1 = A_i^{wt(\mu^-)}\restriction wt(\mu^-)+1$,

i.e., one of the computations converges to the original value. It will then follow that $\Xi_\sigma(x) = \Xi_{\sigma,wt(\xi)}(x) = \Phi_{e,wt(\eta)}(A_i^{wt(\eta)},x)$, and the satisfaction of R_j will follow as in the preceding paragraph.

We conclude the proof of the lemma by verifying (3.1) by induction on $lh(\mu)$. By the construction, $up^2(\xi) = \tau^2$. We compare the locations of $\mu^2 = up^2(\mu^-)$ and τ^2 on T^2. Let $\mu^1 = up(\mu^-)$ and $\xi^1 = up(\xi)$.

Subcase 2.1: $\mu^2 | \tau^2$. Let $\rho^2 = \mu^2 \wedge \tau^2$, and fix β^1 such that $\rho^2 \wedge\langle\beta^1\rangle \subseteq \tau^2$. By (2.4), $\beta^1 \subset \Lambda^1$. First assume that $(\beta^1)^-$ has infinite outcome along β^1. By (2.7), Lemma 2.1, and as $up^2(\xi) = \tau^2$, $\beta^1 \subseteq up(\xi) \subseteq \lambda(\xi)$ and $\beta^1 \subseteq up(\eta) \subseteq \lambda(\eta)$. Hence by (2.6), $\beta^1 \subseteq \lambda(\mu)$. If μ is nonswitching, then (3.1) follows. If μ is switching, then by (2.6), $\mu^1 \supseteq \beta^1$. But then by (2.4) and as $\rho^2 \subset \mu^2$, $\rho^2 \wedge\langle\beta^1\rangle \subseteq \mu^2$, contradicting the definition of ρ^2.

Suppose that $(\beta^1)^-$ has finite outcome along β^1. As $\mu^2 | \tau^2$, there must be a $\tilde{\beta}^1 \subseteq \mu^1$ such that $up((\tilde{\beta}^1)^-) = \rho^2$ and $(\tilde{\beta}^1)^-$ has infinite outcome along $\tilde{\beta}^1$; so if $\beta = out(\tilde{\beta}^1)$, then by (2.5), $\tilde{\beta}^1 = \lambda(\beta)$. By (2.4) and as $\xi \subset \mu$ and $\tilde{\beta}^1 \subseteq \mu^1$, $\xi \subset \beta \subseteq \mu$. As $up^2(\beta^-) = \rho^2 \neq \mu^2 = up^2(\mu^-)$, $\beta \subset \mu$. Hence by (3.1) inductively, $\Phi_{e,wt(\beta)}(A_i^{wt(\beta)},x)\downarrow = \Phi_{e,wt(\beta^-)}(A_i^{wt(\beta^-)},x)\downarrow = \Xi_\sigma(x)$, and $A_i^{wt(\beta)}\restriction wt(\beta^-)+1 = A_i^{wt(\beta^-)}\restriction wt(\beta^-)+1$ for some $i \leq 1$, which we fix. As $\tilde{\beta}^1 \subseteq \mu^1$, it follows from (2.6) that any element entering A_0 or A_1 at any node γ such that $\beta \subset \gamma \subseteq \mu$ is of the form $wt(up(\gamma^-))$ with $up(\gamma^-) \supseteq \tilde{\beta}^1$. Thus by (2.1) and (2.2), $wt(up(\gamma^-)) \geq wt(\tilde{\beta}^1) = wt(\lambda(\beta)) > wt(\beta) > wt(\beta^-)$, so

$$\Phi_{e,wt(\mu)}(A_i^{wt(\mu)},x)\downarrow = \Phi_{e,wt(\mu^-)}(A_i^{wt(\mu^-)},x)\downarrow = \Phi_{e,wt(\beta^-)}(A_i^{wt(\beta^-)},x)\downarrow = \Xi_\sigma(x),$$

and

$$A_i^{wt(\mu)}\restriction wt(\beta^-)+1 = A_i^{wt(\mu^-)}\restriction wt(\beta^-)+1 = A_i^{wt(\beta^-)}\restriction wt(\beta^-)+1.$$

As the computation $\Phi_{e,wt(\beta^-)}(A_i^{wt(\beta^-)},x)\downarrow$ has use $\leq wt(\beta^-)+1$, (3.1) follows.

Subcase 2.2: $\mu^2 \subset \tau^2$. Let $\mu^2\hat{}\langle\overline{\mu}^1\rangle \subseteq \tau^2$. First suppose that $(\overline{\mu}^1)^-$ has infinite outcome along $\overline{\mu}^1$. By (2.4) and (2.8), $\overline{\mu}^1 \subseteq \xi^1,\Lambda^1$, and there is no $\widetilde{\mu}^1 \supset \overline{\mu}^1$ such that $up(\widetilde{\mu}^1) = \mu^2$. By (2.6) and (2.4), μ is nonswitching, so no elements are placed in any set at node μ. Hence (3.1) follows by induction in this case.

Now suppose that $(\overline{\mu}^1)^-$ has finite outcome along $\overline{\mu}^1$. Then μ^2 has infinite outcome along τ^2, so by (2.4), all derivatives of μ^2 along ξ^1 have finite outcome along ξ^1. Hence if μ switches μ^1 (which we assume else (3.1) follows by induction), then $\mu^1 \not\subseteq \xi^1$. Let $\widetilde{\mu}$ be the initial derivative of μ^1 along Λ^0. Then $\xi \subset \widetilde{\mu} \subset \mu$. Hence inductively by (3.1), $\Phi_{e,wt(\widetilde{\mu})}(A_i^{wt(\widetilde{\mu})},x)\downarrow\ = \Xi_\sigma(x)$ with use $\leq wt(\widetilde{\mu})+1$ for some $i \leq 1$, which we fix. Now by (2.6), any element entering A_0 or A_1 at any node γ such that $\widetilde{\mu} \subset \gamma \subseteq \mu$ is of the form $wt(up(\gamma^-))$ with $up(\gamma^-) \supseteq \mu^1$. Thus by (2.1), (2.2), and Lemma 2.1,

$$wt(up(\gamma^-)) \geq wt(\mu^1) = wt(\lambda(\widetilde{\mu})) > wt(\widetilde{\mu}),$$

so

$$A_i^{wt(\mu)} \restriction wt(\widetilde{\mu})+1 = A_i^{wt(\mu^-)} \restriction wt(\widetilde{\mu})+1 = A_i^{wt(\widetilde{\mu}^-)} \restriction wt(\widetilde{\mu})+1.$$

As the computation $\Phi_{e,wt(\widetilde{\mu})}(A_i^{wt(\widetilde{\mu})},x)\downarrow$ has use $\leq wt(\widetilde{\mu})+1$,

$$\Phi_{e,wt(\mu)}(A_i^{wt(\mu)},x)\downarrow\ = \Phi_{e,wt(\mu^-)}(A_i^{wt(\mu^-)},x)\downarrow\ = \Phi_{e,wt(\widetilde{\mu})}(A_i^{wt(\widetilde{\mu})},x)\downarrow\ = \Xi_\sigma(x),$$

and (3.1) follows.

Subcase 2.3: $\mu^2 = \tau^2$. Then μ^- has type 1, so no elements enter any set at μ. (3.1) now follows by induction.

Subcase 2.4: $\mu^2 \supset \tau^2$. Fix $\alpha^1 \subset \Lambda^0$ such that $\tau^2\hat{}\langle\alpha^1\rangle \subseteq \mu^2$. τ^2 cannot have finite outcome along μ^2, else by (2.4), some derivative of τ^2 would have infinite outcome along $\lambda(\mu^-)$ and Case 1 would have been followed. Hence τ^2 has infinite outcome along μ^2. By (2.4), all derivatives of τ^2 along $\lambda(\mu^-)$ have finite outcome along $\lambda(\mu^-)$. As $\mu^2 \supset \tau^2$ and there are no links on T^2, τ^2 is $\lambda^2(\mu^-)$-free, so $\lambda(\mu^-)$-consistent. By (2.11) and Substep 4.2 of Section 2 and as the conditions for Case 1 do not hold, there will be at least (x+1)-many derivatives of τ^2 along $\lambda(\mu^-)$, so there will be such a derivative γ^1 which has the responsibility for argument x, and by the above, γ^1 has finite outcome along $\lambda(\mu^-)$.

Hence if $\delta \subseteq \mu^-$ and $\gamma = \delta^-$ is the principal derivative of γ^1 along μ^-, then $\Phi_{e,wt(\gamma)}(A_0^{wt(\gamma)},x)\downarrow = \Phi_{e,wt(\gamma)}(A_1^{wt(\gamma)},x)\downarrow = \Xi_\sigma(x)$ with use $\leq wt(\gamma)$. By Lemma 2.1 and (2.6), $\lambda(\delta)^- = \gamma^1 \neq \lambda(\delta)$, $\lambda(\delta) \subseteq \lambda(\beta)$ for all β such that $\gamma \subset \beta \subseteq \mu^-$, and any element entering A_0 or A_1 at any β such that $\gamma \subseteq \beta \subseteq \mu^-$ is of the form $wt(up(\beta))$ with $\lambda(\delta) \subseteq up(\beta) \subseteq \lambda(\beta)$. By (2.1) and (2.2), $wt(up(\beta)) \geq wt(\lambda(\delta)) > wt(\delta) > wt(\gamma)$. Hence $\Phi_{e,wt(\mu^-)}(A_0^{wt(\mu^-)},x)\downarrow = \Phi_{e,wt(\mu^-)}(A_1^{wt(\mu^-)},x)\downarrow = \Xi_\sigma(x)$ with use $\leq wt(\gamma)$. At node μ, an element is placed into A_i for at most one $i \leq 1$. Hence by induction, (3.1) must continue to hold for the other value of i. ■

The above lemmas easily yield the following theorem of Lachlan [L] and Yates [Y].

Theorem 3.4 (Minimal Pair): There exist nonrecursive r.e. degrees **a** and **b** such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. ■

4. A Minimal Pair with Non-Low Jumps. In this section, we indicate how to add requirements to construct a minimal pair of r.e. degrees with non-low jumps. These requirements will be needed in Section 5, where we construct a minimal pair of r.e. degrees whose jumps form a minimal pair over **0'**.We note that the type 0 requirements will then be automatically satisfied, and we can either remove them, or choose to leave them in. In order to minimize modifications, we choose to have them remain.

The Requirements: We add to our previous list of requirements the requirements $R^{0,2}_{e,i}$ stating that $A_i' \neq \Phi_e(\emptyset')$ for all $e \in \mathbb{N}$ and $i \leq 1$. Each such requirement is said to have *type 0* and *dimension 2*.

The Basic Module: *Non-lowness requirements.* We satisfy the requirement $R^{0,2}_{e,i}$ by constructing a partial recursive functional Δ_i of two variables which is total on oracle A_i such that for some x, if $|\{u: \Phi_e(\emptyset;u,x)\downarrow = 0\}| = \infty$ then $\lim_v \Delta_i(A_i;v,x) = 1$, if $|\{u: \Phi_e(\emptyset;u,x)\downarrow = 0\}| < \infty$ then $\lim_u \Delta_i(A_i;u,x) = 0$, and such that for all x, $\lim_u \Delta_i(A_i;u,x)\downarrow$. Each such requirement assigned to a node σ of T^2 will be associated with some x. The nodes ν of T^1 derived from σ will be associated with a lower bound $wt(\nu)$ on u, and the nodes η of T^0 derived from σ will be associated with a stage $s = wt(\eta)$ as well, which will provide an upper bound on u. Given x and a node $\eta \in T^0$ working on the requirement for these parameters, we *declare* an axiom $\Delta_{i,s}(A_i^s;v,x) = 0$ with use $u = wt(up(\eta))+1$ for all v

$\leq$ wt(η) for which no current axiom exists, whenever there is no $\tilde{u}$ such that u $\leq \tilde{u} \leq$ wt(η) and $\Phi_{e,s}(\emptyset;\tilde{u},x)\downarrow = 0$. If there is such a $\tilde{u}$, then we place wt(up(η)) $\in A_i$, and redefine $\Delta_{i,s}(A_i^s;v,x) = 1$ with use wt(up(η))+1 for all v $\leq$ wt(η) for which no current axiom exists. There are two possible types of outcomes. If, during the construction, there are only finitely many u such that $\Phi_{e,t}(\emptyset;u,x)\downarrow = 0$, then $\Delta(A_i;v,x) = 0$ for cofinitely many v, and either $\lim_u\Phi_e(\emptyset;u,x)\uparrow$, or $\lim_u\Phi_e(\emptyset;u,x)\downarrow \neq 0$. This outcome will be called the *finite outcome* on T^0 and gives rise to an *infinite outcome* on T^1 and a *finite outcome* on T^2. If we see infinitely many such u, then we place numbers into A_i whenever such u are discovered, allowing us to reset $\Delta_i(A_i;v,x) = 1$ for cofinitely many v. Thus if $\lim_u\Phi_e(\emptyset;u,x)\downarrow$, then $\lim_u\Phi_e(\emptyset;u,x) = 0 \neq 1 = \lim_v\Delta_i(A_i;v,x)$. This outcome will be called the *infinite outcome* on T^0 and gives rise to a *finite outcome* on T^1 and an *infinite outcome* on T^2. ■

We note that no changes need to be made in Section 2. Several additions need to be made for the construction for the requirement $R^{0,2}_{e,i}$. If η is a node of T^0 assigned to such a requirement, ν = up(η), and σ = up(ν), then M_η is the sentence $\exists u \leq wt(\eta)(u \geq wt(\nu)\ \&\ \Phi_{e,wt(\eta)}(\emptyset;u,wt(\sigma))\downarrow = 0)$. A new case must also be added to the construction to handle these requirements.

Construction, Case 4: η is associated with the requirement $R^{0,2}_{e,i}$. Let ν = up(η) and σ = up(ν). Let ξ be chosen as in Section 3, depending on the truth of M_η. If M_η is false, set $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};v,wt(\sigma)) = 0$ with use wt(ν)+1 for all v $\leq$ wt(η) such that $\Delta_{i,wt(\eta)}(A_i^{wt(\eta)};v,wt(\sigma))\uparrow$. If M_η is true, place wt(ν) $\in A_i^{wt(\xi)}$ if η is not the initial derivative of up(η) along ξ, and define $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};v,wt(\sigma)) = 1$ with use wt(ν)+1 for all v $\leq$ wt(η) such that either $\Delta_{i,wt(\eta)}(A_i^{wt(\eta)};v,wt(\sigma))\uparrow$, or $\Delta_{i,wt(\eta)}(A_i^{wt(\eta)};v,wt(\sigma))\downarrow$ with use $\geq$ wt(ν)+1.

At the end of each step of the construction, no matter which requirement η is working on, we define some new axioms as follows. For all $\rho \subset \lambda(\eta)$ which is associated with the requirement $R^{0,2}_{e,i}$, and for which there is a v $\leq$ wt(η) such that $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};v,wt(up(\rho)))$ has not yet been defined, we define an axiom with value m determined as follows. We assume, without loss of generality, that ρ is the longest derivative of up(ρ) along $\lambda(\eta)$. We set m = 0 if ρ has infinite outcome along $\lambda(\eta)$, and m = 1 otherwise. The use of this axiom is wt(ρ)+1. ■

We need an additional case for the Recursiveness Lemma.

Lemma 4.1 (Recursiveness Lemma): For all $i \leq 1$, $\Delta_i(A_i)$ is total and recursive in A_i.

Proof: Recall that, since we still follow the end of the construction in Section 3, we define $\Delta_i(A_i;v,x) = 0$ for all v and x for which no value is assigned by the construction, so $\Delta_i(A_i)$ is total. Thus we must recursively enumerate those $\langle v,x\rangle$ for which this will be the case. As λ is one-to-one and recursive and satisfies (2.1)-(2.3), we can recursively determine whether there is a $\sigma \in T^2$ such that $x = wt(\sigma)$ and $R^{0,2}_{e,i}$ is associated with σ for some e. If the answer in *no,* then $\Delta_i(A_i;v,x) = 0$ for all v.

Suppose that such a σ exists, and let $x = wt(\sigma)$. In order for an axiom $\Delta_i(A_i;v,x) = m$ to be declared for some v and m during the main part of the construction, σ must have an initial derivative μ along $\lambda(\eta)$ for some $\eta \subset \Lambda^0$. It follows from (2.11) and Lemma 2.1 that such an η must lie in the Λ^0-block containing $out^2(\sigma)$, or in the next Λ^0-block. If no such η exists, then $\Delta_i(A_i;v,x) = 0$ for all v.

Suppose that such an η exists. By the last paragraph of the construction above, the construction will declare an axiom $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};v,x) = m$ at ξ with use $wt(\xi) \geq x$ as long as $\mu \subseteq \lambda(\xi)$ and $v \leq wt(\xi)$; and if this condition fails, then the construction will never declare an axiom $\Delta_{i,wt(\delta)}(A_i^{wt(\delta)};v,x) = m$ for any $\delta \supseteq \xi$. As the use of any axiom declared after ξ is $\leq v+1$, we can recursively enumerate those v for which no final axiom will be declared by the construction. ■

We now note that no change is required in the proofs of Lemmas 3.2 and 3.3. (Note that in the proof of Lemma 3.3, we never use the fact that $\dim(R^{0,1}_{e,i}) = 1$.) Hence it remains to show that $R^{0,2}_{e,i}$ is satisfied for all e and $i \leq 1$.

Lemma 4.2: For all $i \leq 1$ and e, $R^{0,2}_{e,i}$ is satisfied.

Proof: Fix e and i. By Lemma 2.3, $R^{0,2}_{e,i}$ is assigned to a node $\sigma \subset \Lambda^2$. Let $x = wt(\sigma)$. First assume that σ has finite outcome along Λ^2. By (2.4) and Lemmas 2.1(v) and 2.2, σ has a principal derivative $\pi \subset \Lambda^1$ which has infinite outcome along Λ^1, and π has infinitely many free derivatives along Λ^0, all of which have finite outcome along Λ^0. By Lemma 2.1, fix $\eta \subset \Lambda^0$ such that η^- is the initial derivative of π along Λ^0. Let μ be the initial derivative of σ along Λ^1. If $\mu \neq \pi$, then $[\mu,\pi]$ will be a primary Λ^1-link. Hence by

(2.10), for all δ such that $\eta \subseteq \delta \subset \Lambda^0$, if $up^2(\delta) = \sigma$, then $up(\delta) \supseteq \pi$; and by (2.8), for all δ such that $\eta \subseteq \delta \subset \Lambda^0$, if $up^2(\delta) = \sigma$, then $up(\delta) \subseteq \pi$. It now follows from the last paragraph of the above construction that for all $v \geq wt(\eta)$, $\Delta_i(A_i;x,v) = 0$ with use $wt(\pi)+1$. Hence $\lim_v \Delta_i(A_i;x,v) = 0$. Now as all derivatives of π have finite outcome along Λ^0, and infinitely many such derivatives exist along Λ^0, there are infinitely many $\delta \subset \Lambda^0$ such that for all u for which $wt(\pi) \leq u \leq wt(\delta)$, $\Phi_{e,wt(\delta)}(\emptyset;u,x) \neq 0$. Thus if $\lim_u \Phi_e(\emptyset;u,x)\downarrow = m$, then $m \neq 0$. We now see that $R^{0,2}_{e,i}$ is satisfied in this case.

Now assume that σ has infinite outcome along Λ^2. By Lemma 2.2, σ has infinitely many free derivatives along Λ^1, and by (2.4), all derivatives of σ along Λ^1 have finite outcome along Λ^1. Let π be the initial derivative of σ along Λ^1 (so π is also the principal derivative of σ along Λ^1). Suppose that the axiom $\Delta_{i,wt(\xi)}(A_i^{wt(\xi)};x,v) = 0$ is defined at $\xi \subset \Lambda^0$. Then by the end of the construction and Case 4, there is a $\delta \subset \xi$ such that $up^2(\delta) = \sigma$, this axiom has use $wt(up(\delta))+1$, and $up(\delta)$ has infinite outcome along $\lambda(\xi)$. Thus a $\lambda(\xi)$-link $[\pi,up(\delta)]$ is formed. As $\pi \subset \Lambda^1$, no ν such that $\eta \subseteq \nu \subset \Lambda^0$ can switch any $\rho \subset \pi$. Hence by (2.10) and as all derivatives of σ have finite outcome along Λ^1, there must be a ν such that $\eta \subseteq \nu \subset \Lambda^0$ and ν switches $up(\delta)$. But then we place $wt(up(\delta)) \in A_i^{wt(\nu)}$, and redefine the axiom $\Delta_{i,wt(\nu)}(A_i^{wt(\nu)};x,v) = 1$. Thus there can be no axiom $\Delta_i(A_i;x,v) = 0$ for any v. Fix v, and fix the shortest $\tau \subset \Lambda^1$ such that $up(\tau) = \sigma$ and $wt(\tau) > v$. Let τ^0 be the principal derivative of τ along Λ^0, and fix $\mu \subset \Lambda^0$ such that $\mu^- = \tau^0$. Then $\Delta_{i,wt(\mu)}(A_i^{wt(\mu)};x,v) = 1$ with use $\leq wt(\tau)+1$. By (2.6) any element entering A_0 or A_1 at any β such that $\mu \subseteq \beta \subset \Lambda^0$ is of the form $wt(up(\beta))$ with $up(\beta) \supset \tau$, and by (2.1) and (2.2), $wt(up(\beta)) > wt(\tau)$. Hence for all v, $\Delta_i(A_i;x,v) = \Delta_{i,wt(\mu)}(A_i^{wt(\mu)};x,v) = 1$. Now infinitely many derivatives of σ have finite outcome along Λ^1; fix such a derivative $\tilde{\sigma} \subset \Lambda^1$. If μ is the principal derivative of $\tilde{\sigma}$ along Λ^0, then $\Phi_{e,wt(\mu)}(\emptyset;x,u) = 0$ for some $u \geq wt(\tilde{\sigma})$, so $\Phi_e(\emptyset;x,u) = 0$. Hence there must be infinitely many u such that $\Phi_e(\emptyset;x,u) = 0$, so $\lim_u \Phi_e(\emptyset;x,u) \neq 1$, and the lemma follows. ■

Theorem 4.3: There are recursively enumerable degrees **a** and **b** such that **a'**,**b'** > **0'** and $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Proof: Immediate from Lemmas 3.1-3.3, 4.1, and 4.2. ■

5. A Minimal Pair Whose Jumps Form a Minimal Pair over 0'. In this section, we show how to construct r.e. degrees **a** and **b** such that **a',b' > 0'**, **a∧b = 0**, and **a'∧b' = 0'**. This construction will require us to start at a higher level tree, T^3, and also to use a form of *backtracking,* a notion introduced in [LL2]. The backtracking machinery was developed to allow nodes to influence the construction even when they do not lie on the current approximation to the true path on all trees. Our proof will use some of the lemmas from [LL2] without proof, so will not be self-contained. We begin by introducing the additional requirements which will be imposed on the construction.

The requirements: We add to our previous list of requirements the requirements $R_e^{1,3}$ stating that there is a recursive functional Ξ such that if $\lim_u\Phi_e(A_0;u,x) = \lim_u\Phi_e(A_1;u,x)$ is total for all x, then $\lim_u\Phi_e(A_0;u,x) = \lim_v\Xi(\emptyset;v,x)$ for all x. Each such requirement is said to have *type 1* and *dimension 3.* By Posner's Lemma [P] and Shoenfield's Limit Lemma [S], it suffices to satisfy these requirements. We note that a different $\Xi = \Xi_\sigma$ is constructed for each $\sigma \in T^3$ dealing with this requirement, and the requirement will be satisfied by Ξ_σ for that σ which lies along the true path. ■

The Basic Module: *Minimal pair requirements over 0'.* The basic module for $R_e^{1,3}$ will be a finite tree consisting of two comparable nodes, of types (1,0) and (1,1). One finite tree for each requirement will begin along each infinite path through the top tree T^3, and will continue until we have completed a path through this finite tree. The first node will have dimension 2, and the second will have dimension 3. The node of dimension 2 will be responsible for checking if the functional Φ_e is total on oracles A_0 and A_1 (checking for totality is a Π^2-sentence), while the second node will try to satisfy the requirement under the assumption that the functionals are total on these oracles, possibly by constructing a recursive functional Ξ. As the two nodes work on requirements of different dimensions, we will separate them on the other trees, but will require that certain subrequirements of the first node always precede a given subrequirement of the second node.

An argument x will be assigned to derivatives of the second node of this module on T^2. Such a derivative will be responsible for showing that $\lim_u\Xi(\emptyset;u,x)$ exists and computes $\lim_v\Phi_e(A_0;v,x)$ whenever this latter limit exists and equals $\lim_v\Phi_e(A_1;v,x)$.

A number u will be assigned to each derivative $\delta \in T^1$ of the first node of the module of T^2 (u will be the weight of the node which begins the subblock containing δ). Such a derivative δ checks to see if $\Phi_e(A_0;v,x)\downarrow$ and $\Phi_e(A_1;v,x)\downarrow$ for all $v,x \le u$. A number u will be assigned to each derivative $\gamma \in T^1$ of the second node of the module of T^2

to which x has been assigned (u will be the weight of the node which begins the subblock containing δ). Such a derivative will try to find numbers $y,z \geq u$ such that $\Phi_e(A_0;y,x)\downarrow \neq \Phi_e(A_1;z,x)\downarrow$. Failing to find such y and z, the node will have the responsibility to define $\Xi(\emptyset;u,x)$.

Suppose that σ is a node of T^0 which is a derivative of a node of type (1,0) to which x and u have been assigned. A number $s = wt(\sigma)$ will be assigned to σ. σ will act as follows. We ask if $\Phi_{e,s}(A_0^s;v,x)\downarrow$ and $\Phi_{e,s}(A_1^s;v,x)\downarrow$ for all $v,x \leq u$. If the answer is *yes*, then we follow the infinite outcome on T^0. This will allow us to protect these computations; no other action is taken. If the answer is *no*, then we follow the finite outcome on T^0. We will now have to deal, on T^0, with other nodes derived from $up(\sigma)$, so make infinitely many attempts to find a *yes* answer to this question. If none is ever found, then either $\Phi_e(A_0;v,x)\uparrow$ or $\Phi_e(A_1;v,x)\uparrow$ for some $v,x \leq u$. Thus $R_e^{1,3}$ will be satisfied, and we will never have to deal with the second node of the module on T^3.

Suppose that τ is a node of T^0 which is a derivative of a node of type (1,1) to which x and u have been assigned. A number $s = wt(\tau)$ will be assigned to τ. Action on T^0 for the module associated with τ is as follows. We ask if there are $y,z \in [u,s]$ such that $\Phi_{e,s}(A_0^s;y,x)\downarrow \neq \Phi_{e,s}(A_1^s;z,x)\downarrow$. If the answer is *yes,* then we follow the infinite outcome on T^0, and will preserve this disagreement as long as the node is derived from a node on the true path of T^1. If this outcome is followed for infinitely many derivatives of $up^2(\tau)$ along the true path of T^1, then $\lim_u\Phi_e(A_0;u,x) \neq \lim_u\Phi_e(A_1;u,x)$, so $R_e^{1,3}$ will be satisfied. If the answer is *no,* then we follow the finite outcome on T^0, and *declare* an axiom $\Xi_s(\emptyset;u,x) = \Phi_{e,s}(A_0^s;u,x)$. We show that if the answer is always *no,* then for all $t \geq s$, either $\Phi_{e,t}(A_0^t;u,x) = \Xi_s(\emptyset;u,x)$ or $\Phi_{e,t}(A_1^t;u,x) = \Xi_s(\emptyset;u,x)$. Hence if we encounter such a requirement for every x on T^1, then if $\lim_u\Phi_e(A_0;u,x)\downarrow = \lim_u\Phi_e(A_1;u,x)\downarrow$ for all x, then $\lim_u\Xi(\emptyset;u,x) = \lim_u\Phi_e(A_0;u,x)$ for all x. ■

It is possible to have $\sigma^2|\tau^2 \in T^2$, both acting for $R_e^{1,3}$, and both derivatives of $\sigma \in T^3$. Furthermore, a derivative σ^1 of σ^2 on T^1 may want to define an axiom $\Xi_\sigma(\emptyset;u,x) = m$, and a derivative τ^1 of τ^2 on T^1 may want to define an axiom $\Xi_\sigma(\emptyset;u,x) = n \neq m$. If we allow this situation to occur when $\sigma^1 \subset \tau^1$, then it will prevent $\lim_u\Xi_\sigma(\emptyset;u,x)$ from being defined. Thus higher priority nodes on T^2 must allow their derivatives to act to capture a disagreement $\Phi_{e,s}(A_0^s;y,x) \neq \Phi_{e,s}(A_1^s;z,x)$ even when they are not on the true path. If, infinitely often, we find σ^1 and τ^1 as above, then such action will ensure that $\lim_u\Phi_e(A_0;u,x) \neq \lim_u\Phi_e(A_1;u,x)$, and hence that Ξ_σ need not be defined.

More precisely, we handle the situation mentioned in the above paragraph as follows. Let $\rho^2 = \sigma^2 \wedge \tau^2$. We will show that the above problem can arise only if $\rho^3 = \mathrm{up}(\rho^2)$ extends an infinite outcome of σ, and σ^1 has infinite outcome along a derivative ρ^1 of ρ^2 which, in turn, has infinite outcome along a derivative τ^1 of τ^2. Instead of declaring an axiom with output n, we act for ρ^1 instead of τ^1 to give it a finite outcome. (The action taken for ρ^1 will allow us to show that limits exist.) This will put all antiderivatives of σ^1 back on the approximation to the true path, and injure at most one of the computations seen by τ^1. Thus σ^1 will see disagreeing computations, and will be able to switch its outcome also. (We call this process a *one-step backtracking* process.) If this happens infinitely often, then σ^2 will have infinite outcome along the true path Λ^2, so σ will have finite outcome along the true path Λ^3, and ρ^3 will not lie on the true path. If this happens only finitely often, then ρ^2 will have sufficiently many derivatives which are free to act according to the dictates of the sentence generating action, and we will be able to show that if ρ^3 and ρ^2 are on the true paths of their respective trees, then the subrequirement associated with ρ^2 is satisfied.

We will need to make sure that σ^1 lies on the approximation to the true path after ρ^1 is switched. Thus whenever σ is on the current approximation to the true path of T^3, and we have encountered a derivative of σ working on argument x earlier in the construction, we will need to have a derivative of σ working on argument x along the current approximation to the true path of T^1. This procedure for ensuring that such derivatives exist is similar to what was done in Substep 4.2 of Section 2, but is carried out for T^2 rather than for T^1. Substep 4.2 also needs to be modified to take the new requirements into account. We now list the revised Substep 4.2 for k = 1, and the new version of this substep for k = 2.

Revisions to Step 4 of Section 2

Step 4: Basic modules are assigned on T^3, rather than on T^2, in the way described for all the individual basic modules. For those modules described in Section 2 which were originally assigned to T^2, finite and infinite outcomes are interchanged on T^3. The process of deciding whether to complete Step 4 in Substep 4.1 requires revision. If k = 0, or if k = 1, $\xi^- = \eta$ and η has finite outcome along ξ, or if k = 2 and η is the initial derivative of $\mathrm{up}(\eta)$, then Step 4 for η is completed in Substep 4.1, and we begin Step 4 for ξ next. Otherwise, we go to Substep 4.2.1 below for ξ if k = 1, to Substep 4.2.2 if k = 2 and η has finite outcome along ξ, and to Substep 4.2.3 if k = 2 and η has infinite outcome along ξ. These substeps add derivatives for minimal pair requirements of dimensions 2 and 3 which seem to be on the true path of the next tree up. On T^1, these derivatives serve to witness that one side of a computation has been injured, when a previous witness has been switched off the true path; this provides a true path computation which prevents derivatives of lower priority nodes on T^2 from destroying the other computation, until the currently destroyed computation recovers. On T^2, this allows us to have free derivatives working on argument x once we have begun to work on x. Thus conflicts between nodes which want to declare different values for an axiom can be resolved by the one step backtracking process described above.

Substep 4.2.1: $k = 1$. We iterate this substep, beginning at $\xi \in T^1$ and assigning requirements to $\gamma \supseteq \xi$, until either we fail to find ρ below, or we reach $\gamma \supset \xi$ such that γ^- has infinite outcome along γ. When we decide, at γ, that this substep will no longer be followed, we return to the beginning of Step 4 for γ. Suppose that either $\gamma = \xi$ or that γ^- has finite outcome along γ. We search for the shortest ξ-consistent $\rho \in T^2$ of type 1 and dimension 2 or 3 such that ρ has a derivative along η, if ρ□ comes from a module for a requirement of dimension 2 then $|\{\mu \subset \gamma\colon \mu \text{ is } \lambda(\xi)\text{-free } \& \ up(\mu) = \rho\}| < wt(\xi)+1$, and if ρ□ comes from a module for a requirement of dimension 3, then there is no derivative of ρ which extends ξ. If such a ρ exists, then the requirement assigned to γ is the requirement which has previously been assigned to the corresponding ρ. (We have the added clause that γ is $\lambda(\xi)$-free which was not required in Section 3, as if T^2 is the last tree, then this property automatically holds.)

Substep 4.2.2: $k = 2$ and η has finite outcome along ξ. We iterate this substep, beginning at $\xi \in T^2$ and assigning requirements to $\gamma \supseteq \xi$, until either we fail to find ρ below, or we reach $\gamma \supset \xi$ such that γ^- has infinite outcome along γ. When we decide, at γ, that this substep will no longer be followed, we return to the beginning of Step 4 for γ. Suppose that either $\gamma = \xi$ or that γ^- has finite outcome along γ. We search for the shortest ξ-consistent $\rho \in T^3$ of type (1,1) and dimension 3 such that ρ has a derivative along η and $|\{\mu \subset \gamma\colon \mu \text{ is } \lambda(\xi)\text{-free } \& \ up(\mu) = \rho\}| \geq z$, where z is determined below and $z \geq wt(\xi)+1$. We choose the smallest possible z to ensure that for each $x \leq wt(\xi)$, there is a $\lambda(\xi)$-free derivative of ρ to which x is assigned. If such a ρ exists, then the requirement assigned to γ is the requirement which has previously been assigned to the corresponding ρ.

Substep 4.2.3: $k = 2$ and η has infinite outcome along ξ. We iterate this substep, beginning at $\xi \in T^2$ and assigning requirements to $\gamma \supseteq \xi$, until either we fail to find ρ below, or we reach $\gamma \supset \xi$ such that γ^- has infinite outcome along γ. When we decide, at γ, that this substep will no longer be followed, we return to the beginning of Step 4 for γ. Suppose that either $\gamma = \xi$ or that γ^- has finite outcome along γ. We search for the shortest ξ-consistent $\rho \in T^3$ of type (1,1) and dimension 3 such that ρ has a derivative along η, and there is an x which is assigned to a derivative of ρ along η, but x is not assigned to a ξ-free derivative of ρ. If such a ρ exists, fix the smallest corresponding x. Then the requirement assigned to γ is the requirement which has previously been assigned to the corresponding ρ, and the argument x is assigned to γ. (For the reader familiar with [LL2], Substeps 4.2.2 and 4.2.3 allow us to carry out a one-step backtracking process.) ■

Lemma 2.1 is still true. Its proof needs to be modified to take the revisions to Substep 4.2 into account, but this is done essentially as before, so we will not present a proof. Lemma 2.2 for n = 3 is also still true, but its proof requires consideration of how derivatives on T^0 of free nodes on T^1 can be restrained by links derived from primary links on T^2. Roughly speaking, one can show that there are infinitely many segments of blocks such that all nodes along them are nonswitching and compute initial segments of the true paths of all trees, and which contain derivatives of all currently consistent nodes. Such a lemma is proved in a general setting in [LL2], and requires a careful analysis of how links are formed. We refer the reader to [LL2] for the required lemmas and the proof of Lemma 2.2. Lemma 2.3 follows exactly as in Section 2.

The construction of Section 3 requires two new cases for the new requirements, in addition to the case added in Section 4. First, however, we need to define the sentence M_η for $\eta \in T^0$ derived from a module of dimension 3 working for $R_e^{1,3}$. If η has type (1,0), let s = wt(η) and let u be the weight of the node which begins the subblock containing up(η); then M_η is the sentence

$$\forall x \le u \forall v \le u(\Phi_{e,s}(A_0^s;v,x)\downarrow \;\&\; \Phi_{e,s}(A_1^s;v,x)\downarrow).$$

If η has type (1,1), let s = wt(η), let u be the weight of the node which begins the subblock containing up(η), and let $x = |\{\delta \subset up^2(\eta): up(\delta) = up^3(\eta) \;\&\; \delta \text{ is } \lambda^2(\eta)\text{-free}\}|$; then M_η is the sentence

$$\exists y \le s \exists z \le s(y \ge u \;\&\; z \ge u \;\&\; \Phi_{e,s}(A_0^s;y,x)\downarrow \ne \Phi_{e,s}(A_1^s;z,x)\downarrow).$$

Revisions to the Construction of Section 3

Case 5: η has type (1,0) and is derived from a module of dimension 3. No action is taken.

Case 6: η has type (1,1) and dimension 3. Let $\sigma = \mathrm{up}^3(\eta)$. If M_η is true, then no action is taken. Suppose that M_η is false. Fix the argument x which is assigned to $\mathrm{up}^2(\eta)$.

Search for the longest $\widehat{\sigma} \subset \mathrm{up}(\eta)$, if any, such that if μ□ is the initial derivative of $\widehat{\sigma}$ along Λ^0, then:

(5.1) $\mathrm{up}^3(\widehat{\sigma}) = \sigma$, the argument x is assigned to $\mathrm{up}(\widehat{\sigma})$, $\widehat{\sigma}$ has type (1,1) and has infinite outcome along $\mathrm{up}(\eta)$, $\Phi_{e,wt(\eta)}(A_0^{wt(\eta)};u,x) \neq \Phi_{e,wt(\mu)}(A_0^{wt(\mu)};v,x)$, where u and v are the parameters assigned to η□ and $\widehat{\sigma}$ respectively; and

(5.2) Either $\widehat{\sigma}$ is $\lambda(\eta)$-free, or there is a $\lambda(\eta)$-link $[\nu,\pi]$ which restrains $\widehat{\sigma}$ and such that $\widehat{\sigma}$ is π-free and π□ is $\lambda(\eta)$-free.

(It will suffice, in (5.1), to look at computations from A_0 alone, as these will previously have been found equal to the corresponding computations from A_1, and neither will have been destroyed. Furthermore, if there is an inequality in the limit, then we will be able to find one for u and v as in (5.1), and as both the A_0 and A_1 computations will agree, we will be able to preserve an inequality through a one-step backtracking process in which at most one of A_0 or A_1 receives a new element.) If no $\widehat{\sigma}$ satisfying (5.1) exists, let $\widehat{\sigma} = \mathrm{up}(\eta)$. Let $m = \Phi_{e,wt(\mu)}(A_0^{wt(\mu)};v,x)$. (At μ, we have begun to define Ξ_σ on argument x, and this definition corresponds to some node σ^2 which may lie on the true path of T^2. We will continue to define axioms for argument x in such a way to ensure that the desired limit will exist. If there are conflicting axioms, then we will return $\sigma^2 = \mathrm{up}(\widehat{\sigma})$ to the true path, and take advantage of the conflicting axioms to find a finite outcome for a new derivative of σ^2.) Define $\Xi_{\sigma,s}(\emptyset;z,x) = m$ with use 1 for all $z \leq u$ such that $\Xi_{\sigma,t}(\emptyset;z,x)\uparrow$ for all $t < s$. (This definition will ensure the existence of limit = m, unless we are able to force a disagreement below. If infinitely many disagreements are forced, then limits need not exist.)

This case is now complete unless $\widehat{\sigma} \neq \mathrm{up}(\eta)$; so suppose that this condition holds, and fix the longest π satisfying (5.2) if it exists. If no such π exists, set $\widehat{\eta} = \eta$□. Suppose that π exists. If $[\nu,\pi]$ is a primary $\lambda(\eta)$-link, let $\widetilde{\eta}$ be a nonswitching extension of η such that $\mathrm{up}(\widetilde{\eta}) = \pi$, and let $\widehat{\eta}$ be the node such that $\widehat{\eta}^- = \widetilde{\eta}$ and $\widehat{\eta}$ switches π. Otherwise (i.e., if the link is not primary), let $\widetilde{\eta}$ be a nonswitching extension of η such that $\mathrm{up}^2(\widetilde{\eta}) = \mathrm{up}(\pi)$, and let $\widehat{\eta}$ be the node such that $\widehat{\eta}^- = \widetilde{\eta}$ and $\widehat{\eta}$ switches $\mathrm{up}(\pi)$. (Note that the process of finding $\widetilde{\eta}$ requires successively taking nonswitching extensions until we have a derivative of up(π) (up^2(π), resp.).)

Let $\tilde{\xi}$ be a nonswitching extension of $\hat{\eta}$ such that up($\tilde{\xi}$) = $\hat{\sigma}$ (as $\hat{\sigma}$ is π-free, it will follow from [LL2: Lemmas 3.5 and 4.5] that such a $\tilde{\xi}$ will exist), and let $\hat{\xi}$ be the node such that $\hat{\xi}^- = \tilde{\xi}$ and $\tilde{\xi}$ has infinite outcome along $\hat{\xi}$. We specify that $\hat{\xi} \subset \Lambda^0$. The process of determining $\hat{\xi}$ is called *backtracking*. Action taken to place elements in sets for all nodes δ of type 0 between η and $\hat{\xi}$ is carried out as in the previous cases, according to the outcome specified by $\hat{\xi}$ for δ rather than according to the truth of the sentence M_δ. ■

Requirements introduced in previous sections will be satisfied. The proofs are virtually the same as those given earlier. We need to show, however, that the backtracking process never determines the outcome of the principal derivative of a node all of whose antiderivatives lie along the true path, and if such a node has infinitely many free derivatives along the true path, then the backtracking process never determines outcomes for any of these nodes.

Lemma 5.1: Fix $\mu \subset \Lambda^0$ such that up^j(μ) is Λ^i-free for all $i \le 3$. Then the outcome of μ is determined by the validity of M_μ.

Proof: Suppose that the outcome for μ is determined by backtracking η to $\hat{\xi}$, i.e., $\eta \subseteq \mu \subset \hat{\xi}$. Let σ = up^3(η). (We will show that there is a $\nu \subseteq \eta$ such that $[\nu,\hat{\xi}^-]$ is a primary $\hat{\xi}$-link, so all nodes μ of T^0 whose outcomes are determined in the process of backtracking from η to $\hat{\xi}^-$ are restrained by a Λ^0-link ending at $\hat{\xi}^-$, hence are not free. Thus we will only need to verify the Lemma for $\mu = \hat{\xi}^-$.) If $\eta \subset \delta \subseteq \tilde{\eta}$, then δ does not switch a node of T^1. Hence as an initial derivative of $\lambda(\eta)$ cannot place any elements into A_0 or A_1, no elements enter A_0 or A_1 for such δ. The construction can place an element into at most one of A_0 or A_1 at $\hat{\eta}$. If $\hat{\eta} \subset \delta \subseteq \tilde{\xi}$, then δ does not switch a node of T^1. Hence as an initial derivative of $\lambda(\eta)$ cannot place any elements into A_0 or A_1, no elements enter A_0 or A_1 for such δ. Now a minimal pair requirement is assigned to $\tilde{\xi}$, so no element enters A_0 or A_1 at $\tilde{\xi}$. Thus the backtracking process from η to $\hat{\xi}$ allows elements $\le$ wt(η) to enter only one of A_0 or A_1. Also, if σ^1 = up($\tilde{\xi}$) works for u = wt(up($\tilde{\xi}$)) and x, then a minimal pair requirement is assigned to σ^1. Hence no element enters A_0 or A_1 at any derivative of σ^1. As $\sigma^1 \subset \lambda(\eta)$, it follows from (2.6) that all elements entering A_0 or A_1 at any node between the initial derivative ν of σ^1 along Λ^0 and η are of the form wt(β□) for some $\beta \supset$

σ^1. Hence as was the case for η□, the computations seen at ν cannot be injured. Let $s = wt(\eta)$, $t = wt(\nu)$, $r = wt(\tilde{\xi})$, and let y be the weight of the node determining the subblock containing $up(\eta)$. Then either

$$\Phi_{e,r}(A_0^r;u,x) = \Phi_{e,t}(A_0^t;u,x) \neq \Phi_{e,s}(A_1^s;y,x) = \Phi_{e,r}(A_1^r;y,x)$$

or

$$\Phi_{e,r}(A_1^r;u,x) = \Phi_{e,t}(A_1^t;u,x) \neq \Phi_{e,s}(A_0^s;y,x) = \Phi_{e,r}(A_0^r;y,x).$$

As $\sigma^1 \subset up(\eta)$ and u is the weight of the subblock containing σ^1, $y \geq u$ by (2.1). Hence as $\hat{\xi}^-$ has infinite outcome along $\hat{\xi}$, $\hat{\xi}$ specifies an outcome for $\hat{\xi}^-$ in accordance with the validity of its associated sentence. Let ν□ be the initial derivative of $\hat{\xi}^-$ along Λ^0. By Lemma 2.1(iii), $\nu \subseteq \eta$. Thus $[\nu,\hat{\xi}^-]$ is a Λ^0-link which restrains all δ such that $\eta \subseteq \delta \subset \hat{\xi}^-$, and the lemma follows in this case. ■

We now show that, whenever we have nodes wanting to define conflicting axioms for x, then $\hat{\sigma}$ as in (5.1) will exist .

Lemma 5.2: Fix x, $\eta \in T^0$, and $\hat{\sigma} \in T^1$ such that (5.1) holds. Assume that for all $\tilde{\sigma} \subset up(\eta)$ such that $up^3(\tilde{\sigma}) = up^3(\hat{\sigma}) = \sigma$ and x is assigned to $up(\tilde{\sigma})$, if μ and ν are the principal derivatives of $\hat{\sigma}$ and $\tilde{\sigma}$, respectively, along η, and u and v are the parameters assigned to $\hat{\sigma}$ and $\tilde{\sigma}$, respectively, then $\Phi_{e,wt(\mu)}(A_0^{wt(\mu)};u,x) = \Phi_{e,wt(\nu)}(A_0^{wt(\nu)};v,x)$. Then there is a $\bar{\sigma} \subset up(\eta)$ such that (5.1) and (5.2) both hold for $\bar{\sigma}$ in place of $\hat{\sigma}$. Furthermore, if $\hat{\xi}$ is chosen as in Case 6 for $\bar{\sigma}$, then either $up(\eta) \not\subseteq \lambda(\hat{\xi})$ or $up(\eta)$ has finite outcome along $\lambda(\hat{\xi})$.

Proof: There are several cases to consider. Let $\eta^1 = up(\eta)$ and $\eta^2 = up^2(\eta)$.

Case 1: η^1 is not the initial derivative of η^2 along $\lambda(\eta)$. By Lemma 2.2(i), the principal derivative $\bar{\sigma}$ of η^2 along $\lambda(\eta)$ is $\lambda(\eta)$-free. By Lemma 2.1(i), $\bar{\sigma} \subset \lambda(\eta)$. Hence by hypothesis, $\bar{\sigma}$ satisfies (5.1) and (5.2).

Case 2: η^1 is the initial derivative of η^2 along $\lambda(\eta)$. Let η^0 be the initial derivative of η^1 along η. We first show that $\eta = \eta^0$. For suppose otherwise. Fix the longest $\widehat{\sigma} \subset \mathrm{up}(\eta)$ which satisfies (5.1) and is the principal derivative of $\mathrm{up}(\widehat{\sigma})$ along $\lambda(\eta)$. Let $s = \mathrm{wt}(\eta^0)$, $t = \mathrm{wt}(\eta)$, and let u be the weight of the node α determining the subblock in which η^1 lies. By the construction, $x \le \mathrm{lh}(\alpha) \le \mathrm{wt}(\alpha) = u$. Furthermore, whenever a node of type (1,1) on T^2 is α-consistent, the corresponding node of T^2 of type (1,0) and in the same module must also be α-consistent. Hence by Substep 4.2.1, $\Phi_{e,s}(A_0^s;x,u)\downarrow = \Phi_{e,s}(A_1^s;x,u)\downarrow$. As $\eta^1 \subset \lambda(\eta)$ and as no elements enter sets at a node dealing with a requirement of type 1, any element entering A_0 or A_1 at any δ such that $\eta^0 \subseteq \delta \subseteq \eta$ is of the form $\mathrm{wt}(\kappa)$ for some $\kappa \supset \eta^1$. By (2.1), (2.2) and Lemma 2.1, for any such κ, $\mathrm{wt}(\kappa) > u$. Hence $\Phi_{e,t}(A_0^t;x,u)\downarrow = \Phi_{e,t}(A_1^t;x,u)\downarrow = \Phi_{e,s}(A_1^s;x,u)\downarrow$. But then by hypothesis (setting $\eta^1 = \widetilde{\sigma}$), it must be the case that $\eta = \eta^0$.

As η^1 is the initial derivative of η^2 along $\lambda(\eta)$ and $\widehat{\sigma}^2$ has a derivative $\widehat{\sigma} \subset \eta^1$, it follows from (2.4) that $\mathrm{up}(\widehat{\sigma}) \not\supseteq \eta^2$. Also, by Lemma 2.1(i), $\eta^2 = \lambda^2(\eta)$. We consider the remaining possibilities for the relative location of $\widehat{\sigma}^2$ and η^2 on T^2.

Case 2.1: $\widehat{\sigma}^2 \subset \eta^2$. $\widehat{\sigma}^2$ cannot be η^2-free, else x would not be assigned to both $\widehat{\sigma}^2$ and η^2. As all links on T^2 are primary, $\widehat{\sigma}^2$ must be restrained by a primary η^2-link. Let $\sigma = \mathrm{up}(\widehat{\sigma}^2)$, and note that $\sigma = \mathrm{up}(\eta^2)$. It now follows from (2.6) and (2.10) that σ is τ^2-consistent for all τ^2 such that $\widehat{\sigma}^2 \subseteq \tau^2 \subseteq \eta^2$. Hence by Substep 4.2.3, if $[\mu^2,\pi^2]$ and $[\widetilde{\mu}^2,\widetilde{\pi}^2]$ are two primary η^2-links restraining $\widehat{\sigma}^2$ and $\pi^2 \subset \widetilde{\pi}^2$, then there is a $\widetilde{\sigma}^2$ such that $\mathrm{up}(\widetilde{\sigma}^2) = \sigma$, x is assigned to $\widetilde{\sigma}^2$, and $\pi^2 \subset \widetilde{\sigma}^2 \subset \widetilde{\pi}^2$. By Lemma 2.1(iii), $\widetilde{\sigma}^2$ will have a principal derivative $\widetilde{\sigma}^1$ such that $\widehat{\sigma} \subset \widetilde{\sigma}^1 \subset \lambda(\eta)$. By hypothesis, we would choose $\widetilde{\sigma}^1$ in place of $\widehat{\sigma}$ in Case 2, yielding a contradiction.

We thus see that there is a unique π^2 for which an η^2-link $[\mu^2,\pi^2]$ restrains $\widehat{\sigma}^2$. Hence $\widehat{\sigma}^2$ must be π^2-free. As $[\mu^2,\pi^2]$ is an η^2-link, π^2 is not the initial derivative of $\mathrm{up}(\pi^2)$ along $\lambda(\pi^2)$. By [LL2: Lemma 4.1], links are nested, so π^2 is $\lambda(\eta)$-free. Hence if we take a nonswitching extension of η until we reach the initial derivative π^0 of a new derivative π^1 of π^2 on T^0, and then give π^0 finite outcome at $\widehat{\eta}$, it will follow from (2.4) that $\lambda^3(\widehat{\eta}) = \lambda(\pi^2)$. (For $(\lambda(\widehat{\eta}))^- = \pi^1$, π^1 has finite outcome along $\lambda(\widehat{\eta})$, and π^1 is not the initial derivative of π^2 along $\lambda(\widehat{\eta})$.) Now by Lemma 2.2(i), that $\widehat{\sigma}$ will satisfy both (5.1) and (5.2).

Case 2.2: $\widehat{\sigma}^2 \mid \eta^2$. Let $\rho^2 = \widehat{\sigma}^2 \wedge \eta^2$. Fix α,β such that $\rho^2\hat{}\langle\alpha\rangle \subseteq \widehat{\sigma}^2$ and $\rho^2\hat{}\langle\beta\rangle \subseteq \eta^2$, and note that $\alpha \ne \beta$. We compare the locations of $\rho^3 = \mathrm{up}(\rho^2)$ and $\sigma = \mathrm{up}(\widehat{\sigma}^2) = \mathrm{up}(\eta^2)$ on T^3. (We will obtain a contradiction except in Case 2.2.5 where σ has infinite outcome along ρ^3.)

Case 2.2.1: $\rho^3 \subset \sigma$. Suppose that $\rho^3\frown\langle\gamma\rangle \subseteq \sigma$. Then $out(\rho^3\frown\langle\gamma\rangle) = \gamma$ is contained in every derivative of σ on T^2, and γ^- is the principal derivative of ρ^3 along every derivative of σ on T^2. Hence $\rho^2 = \gamma^-$ and $\alpha = \beta = \gamma$, a contradiction.

Case 2.2.2: $\rho^3 \mid \sigma$. Let $\delta^3 = \rho^3 \wedge \sigma$. Fix $\tilde{\alpha} \neq \tilde{\beta}$ such that $\delta^3\frown\langle\tilde{\beta}\rangle \subseteq \rho^3$ and $\delta^3\frown\langle\tilde{\alpha}\rangle \subseteq \sigma$. Then $\tilde{\beta}^-$ is the principal derivative of δ^3 along ρ^2, but as $\rho^2 \subset \hat{\sigma}^2$, $\tilde{\beta}^-$ is not the principal derivative of δ^3 along $\sigma \supset \delta^3$. By (2.4), $\tilde{\beta}^-$ must have finite outcome along ρ^2. Again by (2.4), $\tilde{\alpha}^-$ is the principal derivative of δ^3 along both $\hat{\sigma}^2$ and η^2, $\tilde{\alpha}^-$ has infinite outcome along $\tilde{\alpha} \subseteq \hat{\sigma}^2,\eta^2$. Hence $\tilde{\alpha} \subseteq \rho^2$. But then by (2.4), $\tilde{\alpha}^-$ would be the principal derivative of δ^3 along ρ^2, a contradiction.

Case 2.2.3: $\rho^3 = \sigma$. As $\alpha \neq \beta$, ρ^2 must have infinite outcome along either $\hat{\sigma}^2$ or η^2, and $up(\hat{\sigma}^2) = up(\eta^2) = up(\rho^2) = \sigma$, contradicting (2.8).

Case 2.2.4: $\sigma\frown\langle\tilde{\alpha}\rangle \subseteq \rho^3$ and $\tilde{\alpha}^-$ has infinite outcome along $\tilde{\alpha}$. By Lemma 2.1(iii) and (2.8), no derivative of σ can extend any derivative of ρ^3, contradicting the assumption that $\hat{\sigma}^2 \supset \rho^2$.

Case 2.2.5: $\sigma\frown\langle\tilde{\alpha}\rangle \subseteq \rho^3$ and $\tilde{\alpha}^-$ has finite outcome along $\tilde{\alpha}$. We first show that ρ^2 is $\lambda^2(\eta)$-free, by contradiction. For assume not, and let $[\mu^2,\pi^2]$ be a $\lambda^2(\eta)$-link restraining ρ^2. By (2.4), the conditions of Case 2.2, and as $\hat{\sigma} \subset \eta^1$, ρ^2 must have finite outcome γ along $\pi^2 \subset \lambda^2(\eta)$, and $\hat{\sigma} \subset \gamma$. By the existence of $\hat{\sigma}$, (2.1) and (2.2), it follows that $x \leq lh(\hat{\sigma}) \leq wt(\hat{\sigma}) \leq wt(\gamma) \leq wt(\rho^2\frown\langle\gamma\rangle)$. As $\sigma \subset \rho^3$, σ is ρ^3-free, so as, by the conditions of Case 2, η^2 is a derivative of σ which extends π^2, σ must be $\kappa\square^2$-consistent for all κ^2 such that $\pi^2 \subseteq \kappa\square^2 \subseteq \eta^2$. Thus it follows from Subcase 4.2.2 that there is a derivative $\tilde{\sigma}^2$ of σ such that $\rho^2 \subset \tilde{\sigma}^2 \subset \pi^2$. As $\tilde{\sigma}^2$ is not $\lambda^2(\eta)$-free, $\tilde{\sigma}^2 \neq \eta^2$. Since $\hat{\sigma}$ is not $\lambda(\eta)$-free, it follows from Lemma 2.1(ii) and (2.4) that the principal derivative of $\tilde{\sigma}^2$ along $\lambda(\eta)$ must extend $\hat{\sigma}$, contrary to the maximality of $lh(\hat{\sigma})$.

As $\hat{\sigma}^2 \not\subseteq \lambda^2(\eta)$, $\hat{\sigma}$ must be restrained by a primary η^1-link $[\mu^1,\pi^1]$, with $up(\mu^1) = up(\pi^1) = \rho^2$, and by [LL2: Lemma 4.3] (the lemma states that if a node π^1 is not restrained by a $\lambda(\eta)$-link, then π^1 is $\lambda(\eta)$-free) and the preceding paragraph, π^1 is $\lambda(\eta)$-free. Now $up^3(\pi^1) = \rho^3 \supset \sigma$, so by (2.10), $\sigma\square$ must be $\lambda^3(\pi^1)$-free. As $\hat{\sigma} \subset \pi^1$, it follows by a weight argument as in the preceding paragraph and by Subcase 4.2.2 that there is a derivative $\tilde{\sigma}^2$

of σ along $\lambda(\pi^1)$ which is $\lambda(\pi^1)$-free. Hence if we choose $\widehat{\eta}$ in Step 6 for π^1, then $\widehat{\eta}$ provides a finite outcome for π^1. As $up(\mu^1) = up(\pi^1)$, π^1 is not an initial derivative of $up(\pi^1)$, so by (2.4), $\lambda^2(\widehat{\eta}) = \lambda(\pi^1)$. By Lemma 2.2, the principal derivative $\widetilde{\sigma}^1$ of $\widetilde{\sigma}^2$ along $\widehat{\eta}$ is $\widehat{\eta}$-free and π^1 is $\lambda(\eta)$-free, so $\widetilde{\sigma}^1$ will satisfy (5.1) and (5.2).

We now note that at $\widehat{\xi}$, we switch the outcome of $\overline{\sigma}$ from infinite to finite. By the above cases, $\overline{\sigma} \subseteq up(\eta)$. Hence the lemma follows. ■

We now show that $R_e^{1,3}$ is satisfied.

Lemma 5.3: Fix e. Suppose that $\lim_u \Phi_e(A_0;u,x)\downarrow = \lim_u \Phi_e(A_1;u,x)\downarrow$ for all x, and that $\Phi_e(A_0)$ and $\Phi_e(A_1)$ are total. Then there is a recursive function Ξ_τ such that for all x, $\lim_v \Xi_\tau(\emptyset;v,x)\downarrow = \lim_u \Phi_e(A_0;u,x)$.

Proof: By Lemma 2.3, there is a node $\sigma \subset \Lambda^3$ such that $R_e^{1,3}$ is assigned to σ of type (1,0). First assume that σ has infinite outcome along Λ^3. By (2.4) and Lemma 2.2, σ has a free principal derivative σ^2 which has finite outcome along Λ^2. By (2.4) and Lemma 2.2, σ^2 has a free principal derivative σ^1 which has infinite outcome along Λ^1. By Lemma 5.1, (2.4) and Lemma 2.2, σ^1 has infinitely many free derivatives along Λ^0 whose outcomes are not determined by the backtracking process. Thus for infinitely many s, there are $x,u \leq wt(\sigma^1)$ such that either $\Phi_{e,s}(A_0^s;u,x)\uparrow$ or $\Phi_{e,s}(A_1^s;u,x)\uparrow$. Hence either $\Phi_e(A_0)$ is not total, or $\Phi_e(A_1)$ is not total. Thus $R_e^{1,3}$ is satisfied in this case.

Next suppose that σ has finite outcome along Λ^3. Then by Lemma 2.3, there is a node $\tau \subset \Lambda^3$ such that $\tau^- = \sigma$, τ has type (1,1) and $R_e^{1,3}$ is assigned to τ. First assume that τ has finite outcome along Λ^3. By (2.4) and Lemma 2.2, τ has a free principal derivative τ^2 which has infinite outcome along Λ^2. By (2.4) and Lemma 2.2, τ^2 has infinitely many free derivatives τ^1 along Λ^1, all of which have finite outcome along Λ^1. By (2.4), Lemma 2.2, and Lemma 5.1, infinitely many such τ^1 have free principal derivatives whose infinite outcome along Λ^0 is determined by the truth of the sentence assigned to this principal derivative. Fix such a τ^1, and its principal derivative τ^0 along Λ^0. Let $s = wt(\tau^0)$, and let τ^1 work for x and u. Then there are y and z such that $u \leq y,z \leq s$ and $\Phi_{e,s}(A_0^s;y,x)\downarrow \neq \Phi_{e,s}(A_1^s;z,x)\downarrow$. Hence both computations have use $\leq$ s. As $\tau^1 \subseteq \lambda(\tau^0)$ and $\tau^1 \subset \Lambda^1$, it follows from (2.6) and the construction that all elements placed in A_0 or A_1 for $\nu \subset \Lambda^0$ such that $\nu \supseteq \tau^0$ are of the form $wt(\rho)$ for some $\rho \supset \tau^1$. By (2.1), (2.2) and Lemma 2.1, for all such ρ, $wt(\rho) > wt(\tau^0) = s$. Thus $A_i|s+1 = A_i^s|s+1$ for all $i \leq 1$, so

$$\Phi_e(A_0;y,x) = \Phi_{e,s}(A_0^s;y,x) \neq \Phi_{e,s}(A_1^s;z,x) = \Phi_e(A_1;z,x).$$

As there are infinitely many τ^1 each giving rise to a different τ^0 which works on a different u, it follows that if $\lim_v\Phi_e(A_i;v,x)\downarrow$ for all $i \leq 1$, then $\lim_v\Phi_e(A_0;v,x) \neq \lim_v\Phi_e(A_1;v,x)$, so $R_e^{1,3}$ is satisfied in this case.

Finally, suppose that τ has infinite outcome along Λ^3. By (2.4) and Lemma 2.2, τ has infinitely many free derivatives along Λ^2, all having finite outcome along Λ^2, and each associated with a different x. Furthermore, each x is associated with one of these free derivatives. Fix x and the free derivative τ^2 of τ along Λ^2 which is associated with x. Now by (2.4) and Lemma 2.2, τ^2 has a free principal derivative which has infinite outcome along Λ^1. Furthermore, by Lemma 5.2, all nodes $\nu^1 \subset \tau^1$ which are working for Φ_e on argument x and are principal derivatives must specify the same value for the axioms they wish to declare. Thus by the construction $\lim_u\Xi_\tau(\emptyset;u,x)\downarrow$, and the value of the limit is that which τ^1 wishes to specify. Furthermore, Ξ_τ is a total recursive function. Now by (2.4), Lemma 2.2, and Lemma 5.1, τ^1 has infinitely many free derivatives along Λ^0 whose outcomes are determined by the truth of their respective sentences. By Substep 4.2 of the construction, there must be a free node $\sigma^1 \subset \tau^1$ such that $\mathrm{up}^3(\sigma^1) = \sigma$, and $\mathrm{up}(\sigma^1)$ is associated with x. By an argument on uses similar to the one in Lemma 5.1, if u is the weight of the node determining the subblock containing τ^1, then any computation $\Phi_{e,s}(A_0^s;u,x)\downarrow$ or $\Phi_{e,s}(A_1^s;u,x)\downarrow$ which is discovered for a node κ^1 on T^1 cannot be destroyed as long as κ^1 still lies on the path of T^1 being computed, as the elements entering A_0 or A_1 after the initial derivative of κ^1 are of the form $\mathrm{wt}(\delta)$ for some $\delta \supset \kappa^1$, and for such δ□, $\mathrm{wt}(\delta)$ is larger than the use of the computation. Now by the choice of outcome for τ, $\Phi_{e,s}(A_0^s;u,x) = \Phi_{e,s}(A_1^s;u,x)$. By Lemma 5.2, each sentence associated with a free derivative of τ^1 must specify the same value for $\Phi_{e,s}(A_0^s;u,x) = \Phi_{e,s}(A_1^s;u,x)$. Hence if $\lim_v\Phi_e(A_0;v,x) = \lim_v\Phi_e(A_1;z,x)$, then $\lim_v\Phi_e(A_0;v,x) = \lim_u\Xi_\tau(\emptyset;u,x)$, and $R_e^{1,3}$ is satisfied ■

As all requirements are satisfied, we have succeeded in proving:

Theorem 5.4: There are r.e. degrees **a** and **b** such that $\mathbf{a}'|\mathbf{b}'$, $\mathbf{a}\wedge\mathbf{b} = \mathbf{0}$, and $\mathbf{a}'\wedge\mathbf{b}' = \mathbf{0}'$. ■

There are many other types of requirements which can be added to get stronger results. For example, the highness requirements of [LL2] can be used to obtain Cooper's result [C] that there is a high minimal pair. The requirements to make sets $high_2$ of [LL2] can be introduced to make **a** and **b** in Theorem 5.3 $high_2$, and so to make **a'** and **b'** high over **0'**. The satisfaction of requirements requires a more careful development of backtracking as in [LL2: Section 5], or an analysis as in [LL1], if one follows the basic modules of [LL2]. It would also seem that one should be able to use the general framework of [LL2] to show that there are r.e. degrees **a** and **b** which are intermediate in the high/low hierarchy such that for all n, $\mathbf{a}^{(\mathbf{n})}$ and $\mathbf{b}^{(\mathbf{n})}$ form a minimal pair over $\mathbf{0}^{(\mathbf{n})}$.

Other results on minimal pairs, such as Cooper's result [C] that every high degree bounds a minimal pair (of high degrees) (see also Shore and Slaman [SS]) can be proved in this framework, but require more machinery to be introduced. One needs to replace the use of the recursion theorem in [SS], or the existence of a function in the high degree which dominates every recursive function [C], by the Ambos-Spies method [A] which replaces the recursion theorem. The backtracking process for this theorem is much more complex than the one presented here, and needs to be carried out over many, not necessarily consecutive, stages. Rather than presenting a proof here, we plan to present, in another paper, a metatheorem on the framework which will allow us to more easily derive this result. One might expect the following results. For all n, and every $high_n$ degree **h**, there are $high_n$ r.e. degrees $\mathbf{a},\mathbf{b} \le \mathbf{h}$ such that for all $j < n$, $\mathbf{a}^{(j)} \wedge \mathbf{b}^{(j)} = \mathbf{0}^{(j)}$; and for every intermediate degree **d**, there are intermediate degrees $\mathbf{a},\mathbf{b} \le \mathbf{d}$ such that for all j, $\mathbf{a}^{(j)} \wedge \mathbf{b}^{(j)} = \mathbf{0}^{(j)}$.

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