LATTICE EMBEDDINGS INTO THE R.E. DEGREES PRESERVING 1

Klaus Ambos-Spies, Steffen Lempp, Manuel Lerman

Mathematisches Institut, Universität Heidelberg,
W-6900 Heidelberg, Germany
g77@dhdurz1.bitnet
Department of Mathematics, University of Wisconsin,
Madison, WI 53706, USA
lempp@math.wisc.edu
Department of Mathematics, University of Connecticut,
Storrs, CT 06269, USA
mlerman@uconnvm.bitnet

Abstract. We show that the two nondistributive five-element lattices, M_5 and N_5, can be embedded into the r.e. degrees preserving the greatest element.

1. Introduction. The characterization of the finite lattices embeddable into the recursively enumerable (r.e.) degrees (possibly with various additional restrictions, such as preserving the least and/or greatest element) is important to recursion theorists for two reasons: On the one hand, it gives insight into the (very complicated) structure of the r.e. degrees. On the other hand, it constitutes a crucial step in determining the decidability of the universal-existential theory of the partial ordering of the r.e. degrees and of the existential theory of the r.e. degrees in the language of lattices (where meet is a ternary relation), possibly with constant symbols for the least and/or greatest element.

Unfortunately, even though substantial progress has been made, the full characterization of the lattices embeddable into the r.e. degrees remains open. Work by Lachlan, Lerman, Thomason, Yates, and others [6, 13, 14] led to a proof of the embeddability of all countable distributive lattices into the r.e. degrees, while Lachlan [7] showed the embeddability of the two nondistributive five-element lattices, M_5 and N_5. Hopes that all finite lattices might embed into the r.e. degrees were dashed by Lachlan and Soare [9], who exhibited the counterexample S_8. The latest word on lattice embeddings into the r.e. degrees is Ambos-Spies and Lerman [3, 4], who isolate sufficient conditions (for

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both embeddability and nonembeddability). It is not known whether these conditions are complementary.

All known lattice embeddings into the r.e. degrees preserve the least element, 0. Preserving the greatest element, 1, turned out to be quite a bit harder. Lachlan [8], and independently Shoenfield and Soare [10], showed the embeddability of the diamond lattice preserving 1, and Ambos-Spies [1] extended this proof to all countable distributive and some nondistributive lattices (the latter all generalizations of N_5).

Here, we show the embeddability of M_5 into the r.e. degrees preserving 1, which is harder since the usual proof for embedding requires infinitary traces. We also reprove the embeddability of N_5 published in Ambos-Spies's thesis [1] but not elsewhere.

Our notation is standard and generally follows Soare [12] with two exceptions. Here, the *use* of a computation $\Phi^X(y)$ is the largest number *actually* used in the computation and is denoted by $\varphi(y)$ (and similarly for other Greek letters). If the oracle is given as the join of two sets then we assume the use function to give the use separately for each set of the join, thus $\Phi^{(X\oplus Y)\restriction(\varphi(z)+1)}(z)$ is the same as $\Phi^{X\restriction(\varphi(z)+1)\oplus Y\restriction(\varphi(z)+1)}(z)$.

2. The Theorems. We consider embeddings of two lattices, M_5 and N_5. Both have five elements (including the least element, 0, and the greatest element, 1) and are nondistributive. M_5 is a modular lattice and contains three pairwise incomparable elements while N_5 is a nonmodular lattice and contains two comparable elements both of which are incomparable to a third element. An embedding of a lattice into the r.e. degrees is said to *preserve the greatest element*, 1, if the image of 1 under the embedding is the complete r.e. degree $\mathbf{0}'$.

The purpose of this paper is to give the proofs of the following two theorems:

Theorem 1. *The modular nondistributive five-element lattice, M_5, can be embedded into the r.e. degrees preserving the greatest element.*

Theorem 2. *The nonmodular nondistributive five-element lattice, N_5, can be embedded into the r.e. degrees preserving the greatest element.*

(In [1], Ambos-Spies also shows the embeddability of several other lattices (similar to N_5) preserving 1.)

The proofs of the two above theorems are fairly unrelated. We begin with the first and more complicated proof.

3. The Requirements and the Intuition for M_5. We need to construct three incomplete r.e. sets A_0, A_1, and A_2 and an r.e. set B such that any two of the degrees of A_0, A_1, and A_2 join to $\mathbf{0}'$ and meet to the degree of B. We thus also build partial recursive (p.r.) functionals Γ_0, Γ_1, and Γ_2 and infinitely many p.r. functionals Δ and p.r.

functions Λ (of which we suppress the indices), and we ensure the following requirements:

$$\mathcal{S}_i : B \leq_{\mathrm{T}} A_i \text{ (for } i < 3\text{)},$$
$$\mathcal{P}_i : \Gamma_i^{A_j \oplus A_k} = K \text{ (where } \{i,j,k\} = \{0,1,2\} \text{ and } j < k\text{)},$$
$$\mathcal{M}_{j,k}^{\Phi} : \Phi^{A_j} = \Phi^{A_k} \text{ total} \implies \exists \Delta(\Delta^B = \Phi^{A_j})$$
$$\text{(for } j < k < 3\text{, all p.r. functionals } \Phi\text{), and}$$
$$\mathcal{N}_i^{\Psi} : \Psi^{A_i} = K \implies \exists \Lambda(\Lambda = K) \text{ (for } i < 3\text{, all p.r. functionals } \Psi\text{)}.$$

(Here K is the complete r.e. set of the halting problem. Notice that we assume Posner's trick (see Soare [12]) for the $\mathcal{M}$-requirements, so we can assume the same p.r. functional Φ for both A_j and A_k.)

The global requirements $\mathcal{S}_i$ are easily met by putting all numbers entering B also into all A_i so as to ensure $B = A_i \cap R_i$ for recursive sets R_i.

The global requirements $\mathcal{P}_i$ are met by ensuring that the functionals $\Gamma_i^{A_j \oplus A_k}$ are total and correctly compute K. (The hard part here will be totality.)

For the local requirements $\mathcal{M}_{j,k}^{\Phi}$, we use Fejer's strategy [5]. Whenever $\Delta^B(x)$ is defined but equal neither to $\Phi^{A_j}(x)$ nor to $\Phi^{A_k}(x)$ then that strategy puts a number $y \leq \delta(x)$ into B to allow the correction of $\Delta^B(x)$.

For the local requirements $\mathcal{N}_i^{\Psi}$, the problem in meeting $\Psi^{A_0} \neq K$ (setting $i = 0$ to simplify notation) is that protecting computations $\Psi^{A_0}(n)$ for the Sacks preservation strategy conflicts with higher-priority $\mathcal{P}_j$- (and $\mathcal{M}_{j,k}^{\Phi}$-) requirements putting numbers into A_0 (either directly or via B). The usual way to resolve this conflict with $\mathcal{P}_j$ is to fix a number y_0 (independent of n) and to "lift" uses $\gamma_j(y_0) \leq \psi(n)$ by enumerating $\gamma_j(y_0)$ into A_k (for $k \neq 0$) so that $\gamma_j(y_0) > \psi(n)$ can be achieved without having injured $\Psi^{A_0}(n)$. But in order to "lift" all three $\gamma_j(y_0)$ (for $j = 0, 1, 2$), we need to put numbers into at least two sets, namely A_1 and A_2 (since $\Psi^{A_0}(n)$ must not be injured). If we put numbers into A_1 and A_2 simultaneously, this may injure a higher-priority $\mathcal{M}_{1,2}^{\Phi}$-requirement and cause it to destroy $\Psi^{A_0}(n)$ through the correction process. So we have to put a number into A_1 first, wait for Φ^{A_1} to recover, and then put a number into A_2. While we wait for Φ^{A_1} to recover, $\Psi^{A_0}(n)$ is still unprotected and thus may be destroyed before we can put a number into A_2. If this pattern repeats infinitely often then $\Psi^{A_0}(n)$ is undefined but also $\gamma_0(y_0)$ and $\gamma_2(y_0)$ enter A_1 infinitely often, so $\Gamma_0^{A_1 \oplus A_2}(y_0)$ and $\Gamma_2^{A_0 \oplus A_1}(y_0)$ are undefined, injuring our highest-priority requirement.

We use a trick first used by Ambos-Spies, Lachlan, and Soare in their refutation of the existence of a minimal cupping pair of r.e. degrees [2]. It consists in not using $y = y_0$ at first but some $y = y_1 > y_0$, and then repeating the procedure for $y = y_1 - 1$, $y_1 - 2, \dots, y_0$. We will be able to show that once we have reached $y < y_1$, only a $K \restriction (y+1)$-change can cause the destruction of $\Psi^{A_0}(n)$.

The full strategy σ for an $\mathcal{N}_0^{\Psi}$-requirement thus proceeds intuitively as follows (for a fixed number y_0):

(1) Fix y_1 "big", set $n = 0$.
(2) Wait for $\Psi^{A_0} \restriction (n+1) = K \restriction (n+1)$.

(3) For $y = y_1, y_1 - 1, \ldots, y_0 + 1, y_0$, proceed as follows:

(a) Put $\gamma_0(y)$ and $\gamma_2(y)$ into A_1; if $y < y_1$ then also put $\gamma_1(y+1)$ into A_0. Wait for all higher-priority $\mathcal{M}$-strategies to recover.
(b) Put $\gamma_0(y)$ and $\gamma_1(y)$ into A_2, and put $\gamma_2(y)$ into A_0. If $y > y_0$ then wait for all higher-priority $\mathcal{M}$-strategies to recover.

(4) Define $\Lambda_\sigma(n) = K(n)$, increment n by $+1$, and go to 2.

Our strategy assumes that $K \restriction y_0$ will no longer change; so whenever $K \restriction y_0$ does change, we "reset" σ (thus discarding Λ_σ) and start again at (1) with the same y_0. (This constitutes only finite injury to σ.) Furthermore, while σ is in (3) it may be injured by higher-priority $\mathcal{M}^\Phi_{i,j}$-strategies τ with $\tau\hat{\ }\langle 0\rangle \subseteq \sigma$ (i.e. τ's of which σ assumes the infinite outcome). Before performing (3)(b), σ will check if A_0 or A_2 have changed (on an initial segment to be specified later). Before performing (3)(a) (for $y < y_1$), σ will check if A_0 or A_1 have changed (again on an initial segment to be specified later). If so (in either case), σ will destroy $\Psi^{A_0}(n)$ (by putting some $\delta(y) \leq \psi(n)$ into B and thus also into A_0), increment y_1 by $+1$, and go back to (2).

The possible outcomes of the $\mathcal{N}^\Psi_0$-strategy σ (neglecting the finite injury by $K \restriction y_0$) are thus as follows:

(A) σ eventually waits at (2) forever. Then clearly $\Psi^{A_0} \neq K$.
(B) Λ_σ is total. Then we will be able to show $\Lambda_\sigma = K$, a recursive computation for the nonrecursive set K. Thus this outcome cannot actually occur.
(C) Otherwise. Then n must come to a limit, n_0, say; y_1 is incremented infinitely often; and $\Psi^{A_0}(n_0)$ must be destroyed infinitely often (we call this the "infinite outcome for n_0"). We will be able to show in this case that for each y', eventually $\Psi^{A_0}(n_0)$ will always be destroyed before we put $\gamma_j(y')$ into any A_k (for $j, k \in \{0,1,2\}$), thus allowing each Γ_j to be total. In order to show this, we will use the fact that when $y_1 > y'$ and $n = n_0$ then $\Psi^{A_0}(n_0)$ can only be destroyed if either $y \geq y'$ or $K \restriction (y'+1)$ changes (where the latter, of course, can occur at most finitely often for each y'). The hard part will be to show that no higher-priority $\mathcal{M}$-strategy τ with $\tau\hat{\ }\langle 0\rangle \subseteq \sigma$ (i.e. of which σ assumes the infinite outcome) will injure σ infinitely often while $y < y'$. Here we will use the fact that A_0 "holds one side" for τ if τ is an $\mathcal{M}^\Phi_{0,1}$- or $\mathcal{M}^\Phi_{0,2}$-strategy, and that A_j "holds one side" for τ if τ is an $\mathcal{M}^\Phi_{1,2}$-strategy where $j = 2$ between (3)(a) and (3)(b) and $j = 1$ between (3)(b) and (3)(a). (For this, we use a variant of the concept of "configurations" from Slaman's proof of the density of the branching degrees [11].)

We are now ready to describe the full construction.

4. The Construction for M_5. Our tree of strategies is the full binary tree $T = 2^{<\omega}$ with the ordering on T induced by the ordering on 2. The requirements $\mathcal{S}_i$ and $\mathcal{P}_i$ are global and will not be put on the tree. We effectively ω-order the $\mathcal{M}^\Phi_{j,k}$- and $\mathcal{N}^\Psi_i$-requirements as $\{\mathcal{M}_n\}_{n\in\omega}$ and $\{\mathcal{N}_n\}_{n\in\omega}$, respectively. A node $\rho \in T$ works on $\mathcal{M}_n$ if $|\rho| = 2n$ is even, and on $\mathcal{N}_n$ if $|\rho| = 2n+1$ is odd (we call ρ an $\mathcal{M}_n$- or $\mathcal{N}_n$-*strategy*, respectively). We identify 0 with the infinite outcome and 1 with the finite outcome of a strategy ρ.

Each $\mathcal{M}_n$- (or $\mathcal{N}_n$-) strategy $\rho \in T$ builds a p.r. functional Δ_ρ (or a p.r. function Λ_ρ, respectively) to satisfy its requirement. (We will frequently suppress the index on Δ and Λ.)

A strategy $\rho \in T$ is *initialized* by making all its parameters undefined and its functional undefined on all arguments. A strategy $\rho \in T$ is *reset* by initializing it, except that if ρ is an $\mathcal{N}$-strategy then ρ's parameter $y_{0,\rho}$ remains defined. A parameter is defined *big* by setting it to a number greater than any number mentioned thus far in the construction.

We now describe each stage of the construction.

At stage 0, we initialize all strategies and let all $\Gamma_i^{A_j \oplus A_k}$ be undefined on all arguments.

A stage $s+1$ consists of substages $t \leq s+1$ with some additional action before the first and after the last substage. At each substage t, a strategy $\rho \in T$ of length t is "eligible to act" and either "ends the stage" or determines the strategy $\rho' \supset \rho$ eligible to act at substage $t+1$.

Before the first substage of stage $s+1$, we determine the (least) $n \in K_{s+1} - K_s$ (if any). If n exists then, for all i, j, and k such that $\Gamma_i^{A_j \oplus A_K}(n)$ is currently defined, put $\gamma_i(n)$ into B, and thus into $A_{i'}$ for all $i' < 3$, and reset all strategies $\rho \in T$ for which there is an $\mathcal{N}$-strategy $\sigma \leq \rho$ whose parameter $y_{0,\sigma}$ is defined and $> n$. Now proceed to substage 0 of stage $s+1$, at which the strategy $\emptyset$ is eligible to act.

At a substage t of stage $s+1$, suppose ρ is eligible to act. We distinguish cases depending on whether ρ is an $\mathcal{M}$- or an $\mathcal{N}$-strategy.

If ρ is an $\mathcal{M}_{j,k}^{\Phi}$-strategy we first check if there is a (least) x_0 such that $\Delta_\rho^B(x_0)$ is defined but equals neither $\Phi^{A_j}(x_0)$ nor $\Phi^{A_k}(x_0)$. If so then pick $x_1 \leq x_0$ minimal such that

$$\forall x (x_1 \leq x < x_0 \implies \forall i \in \{j,k\} (\Phi^{A_i}(x) \downarrow = \Delta_\rho^B(x) \to \varphi^{A_i}(x) \geq \delta_\rho(x+1))) \,. \tag{1}$$

(I.e. correcting $\Delta_\rho^B(x_0)$ by putting $\delta_\rho(x_0)$ into B and thus all A_i's would trigger a cascade of corrections ending with the correction of $\Delta_\rho^B(x_1)$.) Then put $\delta_\rho(x_1)$ into B and all A_i's.

Next check if the *length of agreement*

$$\ell(\rho) = \max\{x \mid \forall x' < x (\Phi^{A_j}(x') \downarrow = \Phi^{A_k}(x') \downarrow)\}$$

is now greater than at any previous stage at which ρ was eligible to act. If so then for each $x < \ell(\rho)$ (for which now $\Delta_\rho^B(x)$ is undefined) set $\Delta_\rho^B(x) = \Phi^{A_j}(x)$ with the previous use (if $\Delta_\rho^B(x)$ was defined before and no $\delta_\rho(y)$ (for $y \leq x$) has entered B since the last definition of $\Delta_\rho^B(x)$) or with big use (otherwise), and end the substage by letting $\rho\hat{\ }\langle 0\rangle$ be *eligible to act next.* Otherwise, end the substage by letting $\rho\hat{\ }\langle 1\rangle$ be *eligible to act next.*

Now assume an $\mathcal{N}_i^{\Psi}$-strategy ρ is eligible to act at substage t. We describe its action using the flow chart in Diagram 1. After each initialization, ρ starts in state *init*, and at each substage at which it is eligible to act, it proceeds from one state (denoted by a circle) to the next, following the arrows and along the way executing the instructions (in rectangular boxes) and deciding the truth of statements (in diamonds, following the y-arrow iff the statement is true). The parameters defined in the flow chart intuitively have the following meaning: n is the argument at which we currently attempt to define Λ; y is the number for which the $\gamma_{i'}(y)$'s are currently lifted by strategy ρ to a large number; y_0 and y_1 are the current lower and upper bounds for y; i, j, and k are indices of the sets $A_{i'}$ where $\{i, j, k\} = \{0, 1, 2\}$, i is determined by $\mathcal{N}_i^{\Psi}$, and $j < k$; and s_* is the latest stage at which markers were lifted (this parameter is needed to measure (potential) injury).

Given an $\mathcal{M}_{j',k'}^{\Phi}$-strategy τ with $\tau\hat{\ }\langle 0\rangle \subseteq \rho$ (i.e. of which ρ guesses the infinite outcome and by which ρ could be injured), we define (for $l = j'$ or k')

$$m_l^{\tau} = \mu z \geq \max\{\varphi^{A_l}(y) \mid \delta_\tau(y) \leq \psi(n)\}\forall y(\delta_\tau(y)\downarrow\leq z \implies \varphi^{A_l}(y)\downarrow\leq z))\,.$$

(Note that we allow $m_l^{\tau} = \infty$ if $\Phi^{A_l}(y)\uparrow$ for some y with $\Delta_\tau^B(y)\downarrow$.) We say τ *can s_*-injure* ρ at stage $s+1$ if (a) ρ was in *wait*A_j at the beginning of stage $s+1$, $j' = j$, $k' = k$, and some number $\leq m_k^{\tau}[s_*]$ has entered A_k since stage s_* (note that A_k was supposed to "hold one side" for τ); or (b) ρ was in *wait*A_k at the beginning of stage $s+1$, $j' = j$, $k' = k$, and some number $\leq m_j^{\tau}[s_*]$ has entered A_j since stage s_* (note that A_j was supposed to "hold one side" for τ); or (c) $j' \neq j$ or $k' \neq k$, and some number $\leq m_i^{\tau}[s_*]$ has entered A_i since stage s_* (note that A_i was supposed to "hold one side" for τ). If (a), (b), or (c) applies then we define

$$y_\tau = \max\{y \mid \delta_\tau(y)[s_*] \leq \psi(n)[s_*]\}\,.$$

We say ρ *has been s_*-injured* if some number $\leq \psi(n)[s_*]$ has entered A_i since stage s_* (this takes care of miscellaneous injury). Note here that we may assume

$$\forall n(n \leq \psi(n))\,. \tag{2}$$

We end the stage if Diagram 1 specifies so or if $s \leq t$, otherwise we go to substage $t+1$.

At the end of stage $s+1$, i.e. after the last substage, we define $\Gamma_i^{A_j\oplus A_k}(n)$ (for each $\Gamma_i^{A_j\oplus A_k}$ and each $n \leq s$ such that $\Gamma_i^{A_j\oplus A_k}(n)$ is now undefined) with the previous use (if $\Gamma_i^{A_j\oplus A_k}(n)$ was defined before and no $\gamma_i(n')$ (for $n' \leq n$) has entered A_j or A_k since the last definition of $\Gamma_i^{A_j\oplus A_k}(n)$) or with big use (otherwise). Furthermore, we initialize all strategies $>$ the strategy last eligible to act, and proceed to the next stage.

This ends the description of the construction.

5. The Verification for M_5. Our first two lemmas are easy:

Lemma 1 ($\mathcal{S}_i$-Satisfaction Lemma). *$B \leq_{\mathrm{T}} A_i$ for $i = 0, 1,$ and 2.*

Proof. Fix any number x. By the construction, $B \subseteq A_i$. So assume $x \in A_i$, say $x \in A_{i,s}$ for some stage s. But then $x \in B$ iff $x \in B_s$ by the construction. This establishes the claim. ■

Lemma 2 (Resetting Lemma). *If $\rho \in T$ is initialized at most finitely often then it is reset at most finitely often.*

Proof. Since ρ is initialized only finitely often, the same holds for any $\rho' \leq \rho$, and thus $y_{0,\rho}$ comes to a limit. Furthermore, $y_{0,\rho'} \leq y_{0,\rho}$ at any stage at which $y_{0,\rho'}$ is defined for any $\rho' \leq \rho$. Thus ρ is never reset after $y_{0,\rho}$ and $K \upharpoonright \lim_s y_{0,\rho,s}$ settle down. ■

We define the *true path* $f \in [T]$ by induction as follows: Let $\rho = f \upharpoonright n$. Then $f(n) = 0$ if $\rho\hat{\ }\langle 0\rangle$ is eligible to act infinitely often, and $f(n) = 1$ otherwise.

We now turn to the $\mathcal{M}$-requirements:

Lemma 3 ($\mathcal{M}^{\Phi}_{j,k}$-Satisfaction Lemma). *If an $\mathcal{M}^{\Phi}_{j,k}$-strategy $\tau \subset f$ is eligible to act infinitely often and is initialized at most finitely often then it satisfies its requirement.*

Proof. Suppose $\Phi^{A_j} = \Phi^{A_k}$ are both total. By Lemma 2, τ is reset at most finitely often, so Δ_τ is never discarded after some (least) stage s_0, say. By the first part of τ's action in the construction, $\Delta^B_\tau(x)\downarrow \neq \Phi^{A_j}(x)$ is impossible for any x.

Thus we only have to show $\Delta^B_\tau(x)\downarrow$ for all x. Suppose this fails for some (least) x_0, and $\Delta^B_\tau \upharpoonright x_0$ as well as $\Phi^{A_j}(x_0)$ and $\Phi^{A_k}(x_0)$ are defined by correct computations after some (least) stage $s_1 \geq s_0$. Since $\lim_s \ell_s(\tau) = \infty$, we have $\rho\hat{\ }\langle 0\rangle \subset f$ and thus $\Delta^B_\tau(x_0)[s]$ must be defined at infinitely many stages s. Since $\Delta^B_\tau(x_0)\uparrow$, we have $\lim_s \delta_{\tau,s}(x_0) = \infty$. By the way $\delta_{\tau,s}(x_0)$ is defined, it can only be increased by the action of τ or some $\mathcal{N}$-strategy $\sigma \supseteq \tau\hat{\ }\langle 0\rangle$. Once $\delta_{\tau,s}(x_0) \geq \varphi^{A_j}(x_0)$, $\varphi^{A_k}(x_0)$ and $s > s_1$, τ will not increase $\delta_{\tau,s}(x_0)$ by our assumption on s_1 and by (1). There are only finitely many $\mathcal{N}$-strategies $\sigma \supseteq \tau\hat{\ }\langle 0\rangle$ that ever set their $s_* \leq s_1$. Let $s_2 \geq s_1$ be the least stage such that each such σ will either never put $\delta_\tau(y)$ (for $y \leq x_0$) into B after stage s_2 or has already set its $s_* \geq s_1$. Suppose some $\mathcal{N}$-strategy $\sigma \supseteq \tau\hat{\ }\langle 0\rangle$ causes $\delta_\tau(x_0)$ to increase by putting $\delta_\tau(y_\tau)$ into B (for $y_\tau \leq x_0$) at a stage $s > s_2$. By our assumption on s_1 and the minimality of x_0, we have $y_\tau = x_0$. Then $A_j \upharpoonright (m^\tau_j[s_*]+1)$ or $A_k \upharpoonright (m^\tau_k[s_*]+1)$ must have changed between stage s_* and s; without loss of generality assume the former has changed. Since $s_* \geq s_1$ and by our assumption on s_1, $m^\tau_j[s_*] > \varphi^{A_j}(x_0)$. But by the definition of y_τ $(= x_0)$, we have $\delta_\tau(x_0+1)[s_*] > \psi(n)[s_*]$, and thus, by $\delta_\tau(x_0)[s_*] \geq \varphi^{A_j}(x_0)$, we have $m^\tau_j[s_*] \leq \varphi^{A_j}(x_0)$, a contradiction. ■

We now prove a very technical lemma, which constitutes the inductive step in the proofs of the satisfaction of both the $\mathcal{N}$- and the $\mathcal{P}$-requirements:

Lemma 4 (Configuration Lemma). *Let $\{i, j, k\} = \{0, 1, 2\}$ with $j < k$. Let $\sigma \subset f$ be an $\mathcal{N}^{\Psi}_i$-strategy, and suppose that σ is not initialized or reset after some (least) stage s_0. If σ reaches state waitA_k with parameter y at a stage $s_1 > s_0$, then $\Psi^{A_i}(n)$ will not be destroyed after stage s_1 unless $K \upharpoonright y$ changes. If σ reaches waitΨ, having defined $\Lambda(n)$ for some n at a stage $s_1 > s_0$, then $\Psi^{A_i}(n)$ will not be destroyed after stage s_1.*

Proof. Let $\{i,j,k\} = \{0,1,2\}$ with $j<k$. Let

$\mathcal{T}_1 = \{\tau \ \ \mathcal{M}^{\Phi}_{j,k}\text{-strategy} \mid \tau\hat{\ }\langle 0\rangle \subseteq \sigma \wedge \Phi \text{ p.r. functional}\}$, and
$\mathcal{T}_2 = \{\tau \ \ \mathcal{M}^{\Phi}_{j',k'}\text{-strategy} \mid \tau\hat{\ }\langle 0\rangle \subseteq \sigma \wedge \Phi \text{ p.r. functional} \wedge \ (j \neq j' \text{ or } k \neq k')\}$.

(These are the $\mathcal{M}$-strategies "dangerous" to σ.) We will first note that by reverse induction on $y \in [y_0, y_1]$ the following hold:

(3) If σ is in $waitA_j$ with $y = y_1$ then

(3a) $\gamma_i(y_1), \gamma_k(y_1) > m^\tau_k[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_1$, and
(3b) $\gamma_i(y_1), \gamma_k(y_1) > m^\tau_i[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_2$.

(4) If σ is in $waitA_k$ with $y = y_1$ then

(4a) $\gamma_i(y_1), \gamma_j(y_1), \gamma_k(y_1) > m^\tau_j[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_1$, and
(4b) $\gamma_i(y_1), \gamma_j(y_1), \gamma_k(y_1) > m^\tau_i[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_2$.

(5) If σ is in $waitA_j$ with $y < y_1$ then

(5a) $\gamma_i(y), \gamma_j(y+1), \gamma_k(y) > m^\tau_k[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_1$, and
(5b) $\gamma_i(y), \gamma_j(y+1), \gamma_k(y) > m^\tau_i[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_2$.

(6) If σ is in $waitA_k$ with $y < y_1$ then

(6a) $\gamma_i(y), \gamma_j(y), \gamma_k(y) > m^\tau_j[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_1$, and
(6b) $\gamma_i(y), \gamma_j(y), \gamma_k(y) > m^\tau_i[s_*], \psi(n)[s_*]$ for all $\tau \in \mathcal{T}_2$.

(7) If σ reaches $wait\Psi$ while defining $\Lambda(n)$ for some n at a stage $s > s_0$ then (6a)–(6b) hold for this n and s until σ reaches a new state.

It is not hard to verify (3)–(7), keeping in mind the definition of the Γ-uses, the actions of σ, and the fact that the right-hand sides are defined and finite since all the $\tau \in \mathcal{T}_1 \cup \mathcal{T}_2$ have outcome 0 at stage s_* (i.e. greater length of agreement than at all previous τ-stages).

Note that σ ends the stage when it reaches any state other than $wait\Psi$ and initializes all $\rho > \sigma$ when it defines Λ on some n. By this feature, the definition of the m^τ_l's, our assumption on s_0, and (3)–(7) we have after stage s_0:

(8) After σ reaches $waitA_k$ for some y and n, we have

(8a) $\gamma_i(y), \gamma_j(y), \gamma_k(y) > m^\tau_j, \psi(n)$ for all $\tau \in \mathcal{T}_1$, or
(8b) $\gamma_i(y), \gamma_j(y), \gamma_k(y) > m^\tau_k, \psi(n)$ for all $\tau \in \mathcal{T}_1$; and always
(8c) $\gamma_i(y), \gamma_j(y), \gamma_k(y) > m^\tau_j, \psi(n)$ for all $\tau \in \mathcal{T}_2$

unless $K \upharpoonright y$ changes later.

(9) After σ reaches $wait\Psi$ having defined $\Lambda(n)$, we have (8a)–(8c) for $y = y_0$.

By (8a)–(8c) and (9a)–(9c), we have established the lemma since the $\tau \in \mathcal{T}_1 \cup \mathcal{T}_2$ are the only strategies able to destroy $\Psi^{A_i}(n)$ but are prevented from doing so by m^τ_j or m^τ_k for $\tau \in \mathcal{T}_1$, and by m^τ_i for $\tau \in \mathcal{T}_2$, respectively. ■

The satisfaction of the $\mathcal{N}$-requirements now follows easily:

Lemma 5 ($\mathcal{N}_i^\Psi$-Satisfaction Lemma). *If an $\mathcal{N}_i^\Psi$-strategy $\sigma \subset f$ is eligible to act infinitely often and is initialized at most finitely often then it defines $\Lambda_\sigma = K \upharpoonright \operatorname{dom} \Lambda_\sigma$ correctly and satisfies its requirement.*

Proof. Suppose σ is not initialized or reset after some (least) stage s', using Lemma 2. Then, after stage s', σ will pick a big n (call it n_0) and will try to define $\Lambda_\sigma(n)$ for all $n \geq n_0$. Once $\Lambda(n)$ is defined, the corresponding $\Psi^{A_i}(n)$ cannot be destroyed by Lemma 4 and our assumption on s', establishing the first half of our claim.

Now suppose $\Psi^{A_i} = K$, and fix $s_n \geq s'$ (for $n \geq n_0$) such that $\Psi^{A_i} \upharpoonright (n+1)$ is never destroyed after stage s_n and such that $K_{s_n} \upharpoonright (n+1) = K \upharpoonright (n+1)$. Then, after stage s_n, σ will enter states *wait*A_j and *wait*A_k with this n and, since $\Psi^{A_i}(n)$ is no longer destroyed, return to state *wait*Ψ only after having defined $\Lambda_\sigma(n)$. Thus $\Lambda_\sigma(n) = K(n)$ for cofinitely many n, establishing the satisfaction of the $\mathcal{N}_i^\Psi$-requirement. ■

Lemma 6 (Initialization/Eligibility Lemma). *Each $\rho \subset f$ is eligible to act infinitely often and is initialized at most finitely often. Thus all $\mathcal{M}_{j,k}^\Phi$- and $\mathcal{N}_i^\Psi$-requirements are satisfied.*

Proof. Since K is not recursive the domain of Λ_σ for any $\mathcal{N}$-strategy $\sigma \subset f$ must be finite. Thus no $\mathcal{N}$-strategy $\sigma \subset f$ will initialize $\sigma\hat{\ }\langle 0\rangle$ infinitely often. Furthermore, σ can be in states *wait*A_j and *wait*A_k at most finitely often before returning to state *wait*Ψ, and it will return almost always via injury. Thus σ will not end the stage at infinitely many of the stages at which it is eligible to act. This establishes the first half of the lemma. The rest follows by Lemmas 2, 3, and 5. ■

Lemma 7 ($\mathcal{P}_i$-Satisfaction Lemma). *If $\{i,j,k\} = \{0,1,2\}$ and $j < k$ then $\Gamma_i^{A_j \oplus A_k} = K$.*

Proof. By the construction it is impossible to have $\Gamma_i^{A_j \oplus A_k}(z) \downarrow \neq K(z)$ for any z. It thus suffices to show that $\Gamma_i^{A_j \oplus A_k}$ is total. So assume $\Gamma_i^{A_j \oplus A_k}(z)$ is undefined for some (least) z. By the construction, $\Gamma_i^{A_j \oplus A_k}(z)$ is defined infinitely often, and by the assumption on z, $\Gamma_i^{A_j \oplus A_k}(z)$ is destroyed infinitely often by some $\mathcal{N}$-strategy σ. By the way y_0 is picked, only finitely many $\mathcal{N}$-strategies σ can destroy $\Gamma_i^{A_j \oplus A_k}(z)$, so say σ_0 is the $<$-least of them destroying $\Gamma_i^{A_j \oplus A_k}(z)$ infinitely often. Then necessarily $\sigma_0 \subset f$ (by initialization). By Lemmas 6 and 2, σ_0 is not initialized or reset after some (least) stage s_0. Thus $\lim_s y_{1,s} = \infty$, say $y_{1,s} > z+1$ for all $s \geq s_1$ (for some $s_1 > s_0$). Then σ_0 must reach *wait*A_k with $y = z+1$ infinitely often, and, by the first sentence of the proof of Lemma 6, almost always with the same n. So $\Psi^{A_i}(n)$ is defined infinitely often after stage s_0 when σ_0 reaches *wait*A_k with $y = z+1$ but later destroyed. By Lemma 4, $K \upharpoonright y$ must change every time, a contradiction. ■

This concludes the proof of Theorem 1. We now turn to the proof of Theorem 2.

6. The Requirements and the Strategies for N_5. We have to construct r.e. sets

A_0, A_1, A_2, and B such that for $\mathbf{a}_i = \deg(A_i)$ $(i = 0, 1, 2)$ and $\mathbf{b} = \deg(B)$,

$$\begin{aligned} \mathbf{a}_0 \cup \mathbf{a}_2 &= \mathbf{0}', \\ \mathbf{a}_1 \cap \mathbf{a}_2 &= \mathbf{b}, \\ \mathbf{b} < \mathbf{a}_0 &< \mathbf{a}_1, \text{ and} \\ \mathbf{b} &< \mathbf{a}_2. \end{aligned}$$

We ensure this by the following requirements:

$$\begin{aligned} \mathcal{S} &: B \leq_{\mathrm{T}} A_0 \leq_{\mathrm{T}} A_1 \text{ and } B \leq_{\mathrm{T}} A_2, \\ \mathcal{P} &: K \leq_{\mathrm{T}} A_0 \oplus A_2, \\ \mathcal{M}^\Phi &: \Phi^{A_1} = \Phi^{A_2} \text{ total} \implies \exists \Delta(\Delta^B = \Phi^{A_1}) \text{ (for all p.r. functionals } \Phi), \\ \mathcal{D}_i^\Psi &: A_i \neq \Psi^B \text{ (for } i = 0, 2 \text{ and all p.r. functionals } \Psi), \text{ and} \\ \mathcal{D}_1^\Psi &: A_1 \neq \Psi^{A_0} \text{ (for all p.r. functionals } \Psi). \end{aligned}$$

The global requirement $\mathcal{S}$ is met by direct coding, i.e. whenever a number enters B (A_0, resp.) it also enters A_0, A_1, and A_2 (A_1, respectively).

To satisfy the global requirement $\mathcal{P}$ we construct a functional Γ which computes the complete set K from $A_0 \oplus A_2$. We will define Γ implicitly by a marker function $\gamma(x)$ which may be viewed as the use function of Γ. The xth position of γ at the end of stage s will be denoted by $\gamma(x)[s]$. The marker obeys the following rules (for any numbers x, y, s, t):

$$x \neq y \implies \gamma(x)[s] \neq \gamma(y)[t], \tag{γ^0}$$

$$\gamma(x)[s] \neq \gamma(x)[s+1] \implies \gamma(x)[s] < \gamma(x)[s+1] \text{ and, for some } i \in \{0, 2\},\ \gamma(x)[s] \in A_i[s+1], \tag{γ^1}$$

$$\lim_s \gamma(x)[s] \text{ exists}, \tag{γ^2}$$

$$\gamma(x)[s] \notin (A_0 \cup A_2)[s], \text{ and} \tag{γ^3}$$

$$x \in K_{s+1} - K_s \implies \gamma(x)[s] \in (A_0 \cup A_2)[s+1]. \tag{γ^4}$$

Then, by (γ^1) and (γ^2), $\gamma^*(x) := \lim_s \gamma(x)[s]$ exists and $\gamma^*(x) = \sup_s \gamma(x)[s]$. Moreover, by (γ^1) and (γ^3), $\gamma^* \leq_{\mathrm{T}} A_0 \oplus A_2$. Finally, by (γ^3) and (γ^4),

$$\gamma^*(x) = \gamma(x)[s] \implies K_s(x) = K(x).$$

So to compute $K(x)$ from $A_0 \oplus A_2$, $\Gamma^{A_0 \oplus A_2}$ computes the first stage s such that $\gamma^*(x) = \gamma(x)[s]$ and checks whether x has entered K by the end of this stage. If so, $x \in K$; otherwise $x \notin K$.

For the local requirements $\mathcal{M}^\Phi$, as in the preceding proof, we use Fejer's strategy [5]: Whenever $\Delta^B(x)$ is defined but equal neither to $\Phi^{A_1}(x)$ nor to $\Phi^{A_2}(x)$ then the strategy puts a number $y \leq \delta(x)$ into B to allow the correction of $\Delta^B(x)$.

For the local requirements $\mathcal{D}_i^\Psi$ for $i = 0$ ($i = 2$, respectively) we basically use the Friedberg-Muchnik strategy: The strategy has a follower x. If $\Psi^B(x) = 0$ at some stage then it puts x into A_i and tries to preserve the computation $\Psi^B(x)$. The latter conflicts with the $\mathcal{M}$-strategies which are the only ones which put numbers into B. To prevent that an infinitary higher-priority strategy $\mathcal{M}^\Phi$ destroys $\Psi^B(x)$, the $\mathcal{D}_i^\Psi$-strategy attacks only at Φ-expansionary stages and tries to protect the computations $\Phi^{A_2}(y) = \Delta^B(y)$ (or $\Phi^{A_1}(y) = \Delta^B(y)$, respectively). This is achieved by initializing lower-priority $\mathcal{D}$- and $\mathcal{M}$-strategies and by lifting markers $\gamma(z) \le \varphi(y)$ by enumerating them into A_i.

The strategy for the local requirements $\mathcal{D}_1^\Psi$ is similar but slightly more complicated. Again the strategy has a follower x and waits for $\Psi^{A_0}(x) = 0$. Then it wants to put x into A_1 and hold $A_0 \upharpoonright (\psi(x)+1)$ to ensure $A_1(x) \ne \Psi^{A_0}(x)$. Now, to lift a marker $\gamma(z)$, however, we have to put $\gamma(z)$ into A_2, since putting $\gamma(z)$ into A_0 might destroy $\Psi^{A_0}(x)$. So if we put x into A_1 at the same time, for some $\mathcal{M}^\Phi$ and y as above we might destroy both sides of an agreement

$$\Phi^{A_1}(y) = \Delta^B(y) = \Phi^{A_2}(y),$$

thereby causing $\mathcal{M}^\Phi$ to put a number $u \le \delta(y)$ into B and therefore into A_0 (by $\mathcal{P}$), which might destroy $\Psi^{A_0}(x)$. This problem is overcome by doing the attack in two stages.

At the first expansionary stage we lift markers $\gamma(z)$ via A_2 to protect $\Psi^{A_0}(x)$ (and hold A_1 to prevent $\mathcal{M}^\Phi$ from acting). Then, at the next expansionary stage, we put $\gamma(z)$ into A_0 and x into A_1, thereby diagonalizing (and now hold A_2 to prevent $\mathcal{M}^\Phi$ from acting).

7. The Construction for N_5. We define the tree of strategies to be

$$T = \{x \in 2^{<\omega} \mid \forall n(\alpha(2n+1)\downarrow \implies \alpha(2n+1) = 1)\}.$$

Let $\{\mathcal{M}_n\}_{n\in\omega}$ and $\{\mathcal{D}_n\}_{n\in\omega}$ be effective listings of the $\mathcal{M}^\Phi$- and $\mathcal{D}_i^\Psi$-requirements, respectively. As before, node σ works on $\mathcal{M}_n$ if $|\sigma| = 2n$ is even and on $\mathcal{D}_n$ if $|\sigma| = 2n+1$ is odd; and we call σ an $\mathcal{M}_n$- or $\mathcal{D}_n$-strategy, respectively. (Since the $\mathcal{D}_n$-strategies are finitary we have put only their finitary outcome 1 on the tree T.) Every $\mathcal{M}_n$-strategy σ builds a functional Δ_σ to satisfy $\mathcal{M}_n$. Initializing a strategy σ is defined as in the previous construction. We let $\mathrm{In}(\sigma)[s]$ be the greatest stage $t \le s$ at which σ is initialized.

For n with $\mathcal{M}_n = \mathcal{M}^\Phi$ we let

$$\ell(n)[s] = \max\{x : \forall y < x(\Phi^{A_1}(y)[s]\downarrow = \Phi^{A_2}(y)[s]\downarrow)\}\,.$$

Here we adopt the convention that if $\Phi^{A_i}(y)[s]\downarrow \ne \Phi^{A_i}(y)[s+1]$ then $\Phi^{A_i}(y)[s+1]\uparrow$ ("hat-trick", see Soare [12]).

Based on the length function ℓ, σ-stages and σ-expansionary stages are defined as usual by induction on $|\sigma|$: Any stage is a $\emptyset$-stage, and stage s is $\emptyset$-expansionary if $s = 0$, or $s > 0$ and $\forall t < s\ (\ell(0)[t] < \ell(0)[s])$. For σ with $|\sigma| > 0$, s is σ-expansionary if $|\sigma| = 2n$ is even, s is a σ-stage, and

$$\ell(n)[s] > \max\{\ell(n)[t] : t < s \wedge t \text{ is a } \sigma\text{-stage}\}\,.$$

Finally s is a $\sigma\hat{\ }\langle i\rangle$-stage if $|\sigma| < s$ and either $i = 0$ and s is σ-expansionary, or $i = 1$ and s is a σ-stage but not a σ-expansionary stage.

The unique string σ of length s such that s is a σ-stage will be denoted by $\alpha[s]$.

In the following description of the stages of the construction, a number y is called big if y is bigger than all numbers mentioned in the construction up to this point (with the exception of the values of the marker function γ).

Stage 0: Initialize all strategies σ. Let $\gamma(x)[0] = 2\langle x, 0\rangle$.

Stage $s+1$: The stage consists of 6 steps.

Step 1: Initialize all strategies σ with $\alpha[s] < \sigma$.

Step 2 ($\mathcal{D}$-Strategies): For any $\sigma \subseteq \alpha[s]$ with $|\sigma| = 2n+1$ odd and $\mathcal{D}_n = \mathcal{D}_i^{\Psi}$, σ requires attention if either σ has no follower or, for the follower x, $A_i(x)[s] = \Psi^B(x)[s] = 0$ (if $i \in \{0, 2\}$) or $A_1(x)[s] = \Psi^{A_0}(x)[s] = 0$ (if $i = 1$).

Fix the least σ which requires attention, say $|\sigma| = 2n+1$, $\mathcal{D}_n = \mathcal{D}_i^{\Psi}$. (If no σ requires attention, Step 2 is vacuous.) Say that σ acts. Initialize all strategies σ' with $\sigma < \sigma'$. If σ has no follower, let x be the least big odd number and appoint x as a σ-follower. If σ has a follower, say x, then distinguish the following 3 cases.

Case 1: $i \in \{0, 2\}$. Then put x into A_i. Moreover, for any $y \geq \mathrm{In}(\sigma)[s]$, put $\gamma(y)[s]$ into A_i and let $\gamma(y)[s+1] = 2\langle y, s+1\rangle$.

Case 2: $i = 1$ and x is not yet confirmed. Then, for any $y \geq \mathrm{In}(\sigma)[s]$, put $\gamma(y)[s]$ into A_2, and let $\gamma(y)[s+1] = 2\langle y, s+1\rangle$. Say that x is *confirmed.*

Case 3: $i = 1$ and x is confirmed. Then put x into A_1. Moreover, for any $y \geq \mathrm{In}(\sigma)[s]$, put $\gamma(y)[s]$ into A_0 and let $\gamma(y)[s+1] = 2\langle y, s+1\rangle$.

Step 3 ($\mathcal{P}$-Strategy): For any x such that $x \in K_{s+1} - K_s$ and $\gamma(x)[s+1]$ has not been redefined in Step 2, put $\gamma(x)$ into A_0, let $\gamma(x)[s+1] = 2\langle x, s+1\rangle$, and, for any σ such that $|\sigma|$ is odd and $x < \mathrm{In}(\sigma)[s]$, cancel the σ-follower (if there is any).

If not stated otherwise above, $\gamma(y)[s+1] = \gamma(y)[s]$.

Step 4 ($\mathcal{M}$-Strategies; correction): For any σ such that σ has not been initialized in the previous steps and $|\sigma|$ is even, say $|\sigma| = 2n$ and $\mathcal{M}_n = \mathcal{M}^{\Phi}$, and for any number y do the following: If $\Phi^{A_1}(y)[s] \neq \Delta_\sigma^B(y)[s] \downarrow\neq \Phi^{A_2}(y)[s]$ then put $\delta_\sigma(y)[s]$ into B, let $\delta_\sigma(y)[s+1]\uparrow$ and $\Delta_\sigma^B(y)[s+1]\uparrow$, and initialize all σ' with $\sigma\hat{\ }\langle 0\rangle <_L \sigma'$.

Step 5 ($\mathcal{M}$-Strategies; extension): For any σ such that $\sigma\hat{\ }\langle 0\rangle \subseteq \alpha[s]$, σ has not been initialized in the previous steps and such that $|\sigma|$ is even, say $|\sigma| = 2n$ and $\mathcal{M}_n = \mathcal{M}^{\Phi}$, and for any number y do the following: If $y < \ell(n)[s]$ and $\Delta_\sigma^B(y)\uparrow$ then let $\Delta_\sigma^B(y)[s+1] = \Phi^{A_1}(y)[s] = \Phi^{A_2}(y)[s]\downarrow$ and let $\delta_\sigma(y)[s+1]$ be the previous use (if $\Delta_\sigma^B(y)$ has been defined before and no $\delta_\sigma(y')$ for $y' \leq y$ has entered B since the last definition of $\Delta_\sigma^B(y)$) or the least big odd number (otherwise).

Step 6 ($\mathcal{S}$-Strategy): Put any number which has entered B (A_0) in one of the previous steps also into A_0, A_1, and A_2 (A_1, respectively).

This completes the description of the construction.

8. The Verification for N_5.

Lemma 1 ($\mathcal{S}$-Lemma). *$B \leq_{\mathrm{T}} A_0 \leq_{\mathrm{T}} A_1$ and $B \leq_{\mathrm{T}} A_2$.*

Proof. Any number x which enters any set under construction at stage $s+1$ has not entered any other set under construction at any previous stage. So the claim is immediate by Step 6 of stage $s+1$. ■

The *true path* $f \in [T]$ is defined to be the leftmost path through T such that for any n, $f \restriction n \subseteq \alpha[s]$ for infinitely many s.

We say x is a permanent σ-follower if x is σ-follower from some stage on.

Lemma 2 (Initialization Lemma). *Let $\sigma \subset f$.*

(a) *σ is initialized only finitely often.*
(b) *If $|\sigma|$ is odd then σ acts only finitely often and has a permanent follower.*

Proof. We proceed by induction on $|\sigma|$.

Fix s_0 such that $\sigma \leq \alpha[s]$ for all $s \geq s_0$ and such that, by inductive hypothesis, no σ' with $\sigma' \subset \sigma$ acts after stage s_0. Then σ will not be initialized in Step 1 or 2 of any stage $s > s_0$. Moreover, no $\delta_\tau(y)$ for $\tau\hat{\ }\langle 0\rangle <_L \sigma$ will be appointed after stage s_0 (in Step 5), whence, there will be a stage $s_1 > s_0$ such that σ will not be initialized in Step 4 of any stage $s \geq s_1$ and hence will not be initialized after stage s_1 at all.

Now, if $|\sigma|$ is odd, fix $s_2 > s_1$ such that $K_{s_2} \restriction (s_1+1) = K \restriction (s_1+1)$. Then no follower of σ will be cancelled in Step 3 of any stage $s \geq s_2$, whence any σ-follower existing after stage s_2 is permanent. Moreover, if s_3 is the least σ-stage $> s_2$ then either there is a σ-follower at the end of stage s_3 or a σ-follower is appointed at stage s_3. So σ will act at most once (if $i \in \{0,2\}$) or twice (if $i=1$) after stage s_3+1. ■

Lemma 3 ($\mathcal{P}$-Lemma). *$K \leq_{\mathrm{T}} A_0 \oplus A_1$.*

Proof. By the discussion of the $\mathcal{P}$-strategy preceding the construction it suffices to show that the function $\gamma(x)[s]$ satisfies conditions (γ^0)–(γ^4). For (γ^0)–(γ^1) and (γ^3)–(γ^4) this is immediate by the construction. For a proof that $\lim_s \gamma(x)[s]$ exists fix x. By Lemma 2, choose stages s_1 and s_0 such that $s_1 > s_0 > x$, $\alpha[s_0] \subseteq \alpha[s_1] \subset f$, and no σ with $\sigma \leq \alpha[s_0]$ acts after stage s_1. Since any σ with $\alpha[s_0] < \sigma$ is initialized in Step 1 of stage $s_0+1 > x$ and since only such σ will act after stage s_1, $\gamma(x)[s]$ will not be redefined in Step 2 of any stage $s+1 > s_1$. So the value of $\gamma(x)[s]$ will change at most once after stage s_1, namely if x enters K after that stage. ■

Lemma 4 (Δ-Correctness Lemma). *Let $\mathcal{M}^\Phi = \mathcal{M}_n$, $|\sigma| = 2n$, and $\sigma\hat{\ }\langle 0\rangle \subset f$. If s is a $\sigma\hat{\ }\langle 0\rangle$-stage and $\Delta^B_\sigma(y)[s]\downarrow$ then $\Delta^B_\sigma(y)[s] = \Phi^{A_1}(y)[s]$.*

Proof. For a contradiction assume that $\Delta^B_\sigma(y)[s] \neq \Phi^{A_1}(y)[s]$. Let t be the greatest stage $< s$ such that $\Delta^B_\sigma(y)[t]\uparrow$. Then t is a σ-expansionary stage and

$$\Delta^B_\sigma(y)[s] = \Delta^B_\sigma(y)[t+1] = \Phi^{A_1}(y)[t] = \Phi^{A_2}(y)[t]\,.$$

Since s is σ-expansionary, too, we must have $\Delta^B_\sigma(y)[s] \neq \Phi^{A_1}(y)[s]\downarrow$ and $\Delta^B_\sigma(y)[s] \neq \Phi^{A_2}(y)[s]\downarrow$. So, by the "hat-trick", there must be a stage v such that $t < v < s$ (whence

$\Delta_\sigma^B(y)[t+1] = \Delta_\sigma^B(y)[v] = \Delta_\sigma^B(y)[s])$, and $\Delta_\sigma^B(y)[v] \neq \Phi^{A_1}(y)[v]$ and $\Delta_\sigma^B(y)[v] \neq \Phi^{A_2}(y)[v]$ (where one of the right-hand side computations is undefined). So, by Step 4 of the construction, $\Delta_\sigma^B(y)[v+1] \uparrow$ contrary to the choice of t. ■

Lemma 5 ($\mathcal{M}$-Lemma). *Each $\mathcal{M}^\Phi$ is met.*

Proof. Without loss of generality, we may assume that $\Phi^{A_1} = \Phi^{A_2}$ is total. Pick n and σ such that $\mathcal{M}^\Phi = \mathcal{M}_n$, $|\sigma| = 2n$, and $\sigma \subset f$. Then, by assumption, $\lim_s \ell(n)[s] = \infty$. So there are infinitely many σ-expansionary stages, whence $\sigma\,\hat{}\,\langle 0\rangle \subset f$. Moreover, by Lemma 2, there is a stage after which σ is never initialized. It easily follows from Step 5 in the construction that Δ_σ^B is total and, with Lemma 4, that $\Delta_\sigma^B = \Phi^{A_1}$. ■

Lemma 6 ($\mathcal{D}$-Lemma). *Each $\mathcal{D}_i^\Psi$ is met.*

Proof. We give the proof for $i = 1$. (The other cases are similar and somewhat simpler.) Fix n and σ such that $\mathcal{D}_n = \mathcal{D}_i^\Psi$, $|\sigma| = 2n+1$ and $\sigma \subset f$. By Lemma 2 there is a stage s_0 such that at stage s_0+1 a σ-follower x is appointed which will never be cancelled. We will show that $A_1(x) \neq \Psi^{A_0}(x)$. We distinguish two cases.

Case 1. There is a σ-stage $s > s_0$ such that

$$\Psi^{A_0}(x)[s] \downarrow = 0\,.$$

Then let s_1 be the least such stage. By the choice of s_0, σ acts at stage s_1+1 and x becomes confirmed. Now let s_2 be the least σ-stage $> s_1$.

We claim that

$$(*) \qquad B[s_1] \restriction s_1 = B[s_2] \restriction s_1 \text{ and } A_i[s_1] \restriction s_1 = A_i[s_2] \restriction s_1$$

for $i = 0, 1$, whence in particular

$$\Psi^{A_0}(x)[s_2] = \Psi^{A_0}(x)[s_1] = 0$$

via the same computation. For a proof of $(*)$ we note that all strategies $\sigma' > \sigma$ are initialized whence such strategies cannot injure $(*)$. Moreover, since, by choice of s_0, σ is not initialized after this stage, no $\mathcal{M}$-strategy σ' with $\sigma'\,\hat{}\,\langle 0\rangle <_L \sigma$ will put a number into B after stage s_0 and no $\mathcal{D}$-strategy σ' with $\sigma' < \sigma$ will put a number into any set A_j ($j = 0, 1, 2$). Since σ itself does not destroy $(*)$ (at stage s_1+1 it puts numbers into A_2 only and it does not act before stage s_2+1 again), this leaves only the $\mathcal{P}$-strategy and $\mathcal{M}$-strategies τ with $\tau\,\hat{}\,\langle 0\rangle \subseteq \sigma$. Now, by action of σ at stage $s+1$, $\gamma(y)[s_1+1] > s_1$ for all y with $y \geq \mathrm{In}(\sigma)[s_1]$. So if the $\mathcal{P}$-strategy injures $(*)$, then it enumerates some $\gamma(y)[s]$ with $y < \mathrm{In}(\sigma)[s_1] \leq \mathrm{In}(\sigma)[s]$ into some A_i, which will result in cancellation of the follower x contrary to the choice of x. Finally, consider an $\mathcal{M}$-strategy τ with $\tau\,\hat{}\,\langle 0\rangle \subseteq \sigma$. Then s_1 is τ-expansionary and, by Lemma 4,

$$\forall y(\Delta_\tau^B(y)[s_1] \downarrow \Longrightarrow \Delta_\tau^B(y)[s_1] = \Phi^{A_1}(y)[s_1] \downarrow = \Phi^{A_2}(y)[s_1])\,.$$

Now τ will injure $(*)$ at a stage $s+1 > s_1$ only if, for such a number y, $\Phi^{A_1}(y)[s_1] \neq \Phi^{A_1}(y)[s]$ and $\Phi^{A_2}(y)[s_1] \neq \Phi^{A_2}(y)[s]$, i.e. if some other strategy has injured $A_1[s_1] \upharpoonright s_1 = A_1[s_2] \upharpoonright s_1$ *before*. As we have shown, however, this will not happen.

Now at stage s_2+1, σ becomes active again and puts x into A_1. To show that $\Psi^{A_0}(x) = \Psi^{A_0}(x)[s_2] = 0$ (whence $\mathcal{D}_n$ is met) it suffices to show

$$(**) \qquad B[s_2] \upharpoonright s_2 = B \upharpoonright s_2 \text{ and } A_2[s_2] \upharpoonright s_2 = A_2 \upharpoonright s_2 .$$

This is shown as $(*)$. We only have to note that σ acts at stage s_2+1 for the last time and that it does not put any numbers into B or A_2 at this stage. (Also note that the $\gamma(y)[s_2]$ which σ enumerates into A_0 have been lifted at stage s_1 already whence they cannot injure the computation $\Psi^{A_0}(x)$.)

Case 2. Otherwise. Then $\Psi^{A_0}(x) \neq 0$ and x never enters A_1. So $A_1 \neq \Psi^{A_0}$ whence $\mathcal{D}_1^{\Psi}$ is met. ■

This completes the proof of Theorem 2.

References

1. K. Ambos-Spies, *On the Structure of the Recursively Enumerable Degrees*, Doctoral Dissertation, University of Munich, 1980.
2. K. Ambos-Spies, A. H. Lachlan, R. I. Soare, *The continuity of cupping to* $\mathbf{0}'$, Ann. Pure Appl. Logic (to appear).
3. K. Ambos-Spies, M. Lerman, *Lattice embeddings into the recursively enumerable degrees*, J. Symbolic Logic **51** (1986), 257–272.
4. ———, *Lattice embeddings into the recursively enumerable degrees, II*, J. Symbolic Logic **54** (1989), 735–760.
5. P. A. Fejer, *Branching degrees above low degrees*, Trans. Amer. Math. Soc. **273** (1982), 157–180.
6. A. H. Lachlan, *Lower bounds for pairs of recursively enumerable degrees*, Proc. London Math. Soc. **16** (1966), 537–569.
7. ———, *Embedding nondistributive lattices in the recursively enumerable degrees*, Conference in Mathematical Logic, London, 1970, Lecture Notes in Mathematics No. 255, SpringerVerlag, Berlin, Heidelberg, New York, 1972, pp. 149–177.
8. ———, *Decomposition of recursively enumerable degrees*, Proc. Amer. Math. Soc. **79** (1980), 629–634.
9. A. H. Lachlan, R. I. Soare, *Not every finite lattice is embeddable in the recursively enumerable degrees*, Adv. in Math. **37** (1980), 74–82.
10. J. R. Shoenfield, R. I. Soare, *The generalized diamond theorem*, abstract # 219, Recursive Function Theory Newsletter **19** (1978).
11. T. A. Slaman, *The density of infima in the recursively enumerable degrees*, Ann. Pure Appl. Logic **52** (1991), 155–179.
12. R. I. Soare, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Omega Series, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.
13. S. K. Thomason, *Sublattices of the recursively enumerable degrees*, Z. Math. Logik Grundlag. Math. **17** (1971), 272–280.
14. C. E. M. Yates, *A minimal pair of recursively enumerable degrees*, J. Symbolic Logic **31** (1966), 159–168.