LATTICE EMBEDDINGS INTO THE R.E. DEGREES PRESERVING 0 AND 1

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Abstract. We show that a finite distributive lattice can be embedded into the r.e. degrees preserving least and greatest element iff the lattice contains a join-irreducible noncappable element.

1. Introduction. Recursion theory explores the theoretical bounds of computability. A central notion in the field is that of a recursively enumerable set, i.e. a set of natural numbers that can be listed "effectively", e.g. by a Turing machine. Recursively enumerable sets appear in all branches of mathematics, such as the sets of solutions of diophantine equations or in the famous word problem of group theory. The Turing degrees of such sets (as a measure of their computational complexity) have thus been studied extensively, and their structure turned out to be very complex.

The characterization of the finite lattices embeddable into the recursively enumerable (r.e.) degrees (possibly with various additional restrictions, such as preserving least and/or greatest element) has two important aspects to recursion theorists: On the one hand, it gives insight into the very complicated algebraic structure of the r.e. degrees. On the other hand, it is a step towards the decidability of the universal-existential theory of the partial order of the r.e. degrees (for which the lattice embeddings problem gives a partial result); and of the existential theory of the r.e. degrees in the language of lattices (with meet as a ternary relation), possibly with constant symbols for least and/or greatest element.

Lachlan [5] and Yates [10] proved the diamond lattice to be embeddable into the r.e. degrees, and further work by Lachlan, Lerman, and Thomason [9] showed that

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indeed all countable distributive lattices can be embedded into the r.e. degrees. Lachlan [6] further established the embeddability of the two five-element nondistributive lattices into the r.e. degrees but Lachlan and Soare [7] found a nonembeddable eight-element lattice, S_8. The full characterization of the finite lattices embeddable into the r.e. degrees remains open, the latest word being Ambos-Spies and Lerman [3, 4], who isolate very complicated embeddability and nonembeddability conditions not known to be complementary.

All known lattice embeddings preserve the least element, 0. Preserving both 0 and the greatest element, 1, is much harder. While, by the Sacks Density Theorem, any countable totally ordered lattice can be embedded into the r.e. degrees preserving 0 and 1, the same fails even for the diamond lattice by Lachlan's Nondiamond Theorem [5]. Ambos-Spies [1] further showed that any countable disconnected distributive lattice can be embedded into the r.e. degrees preserving 0 and 1. (A lattice is called *disconnected* if it contains a nontrivial totally ordered interval of elements comparable with any other element.)

For a decade, the major open question in this area has been the embeddability of the double diamond lattice into the r.e. degrees preserving 0 and 1. (The double diamond lattice is the seven-element lattice formed by two diamond lattices where the greatest element of the first is equal to the least element of the second diamond lattice.) In an attempt to refute this embeddability, people tried to show that no two incomparable r.e. degrees can have an infimum if exactly one of them is a *cappable* r.e. degree (i.e. half of a minimal pair); and several claims to this were made and later withdrawn.

In this paper, we answer the double diamond embeddability question positively and indeed provide a full characterization of the finite distributive lattices embeddable into the r.e. degrees preserving 0 and 1. They are exactly the finite distributive lattices containing a join-irreducible noncappable element.

Our notation is standard and generally follows Soare [8] with two exceptions: Here, the *use* of a computation $\Phi(X;y)$ is the largest number *actually* used in the computation and denoted by $\varphi(y)$ (or $\varphi(X;y)$) (and similarly for other Greek letters). If the oracle is given as the join of two or more sets then we assume the use function to give the use separately for each set in the join, thus $\Phi((X_1 \oplus \cdots \oplus X_n) \restriction (\varphi(x)+1); y)$ is the same as $\Phi(X_1 \restriction (\varphi(x)+1) \oplus \cdots \oplus X_n \restriction (\varphi(x)+1); y)$.

2. The Theorem. We start with two definitions:

Definition 1. Let $\mathcal{L} = \langle L, \leq, \vee, \wedge, 0, 1\rangle$ be a lattice with (distinct) least and greatest element.

(i) $a \in L$ is called *join-irreducible* if $a \neq 0$ and for no $b, c < a$, $b \vee c = a$.

(ii) $a \in L$ is called *cappable* if there is $b \neq 0$ such that $a \wedge b = 0$. (We thus allow $a = 0$.)

We say a lattice embedding into the r.e. degrees *preserves least and greatest element* (or *preserves* 0 *and* 1, for short) if the images of 0 and 1 under the embedding are the r.e. degrees $\mathbf{0}$ and $\mathbf{0}'$, respectively.

Our main theorem is now the following:

Theorem. *A finite distributive lattice is embeddable into the r.e. degrees preserving least and greatest element iff the lattice contains a join-irreducible noncappable element.*

Proof. We show the easy direction here and defer the other direction to the remaining sections of this paper.

Suppose every join-irreducible element of the lattice $\mathcal{L}$ is cappable. But 1 is the join of all join-irreducible elements of $\mathcal{L}$ (see e.g. Fact (2) in Section 3). Thus the image of 1 is the join of finitely many cappable r.e. degrees and itself cappable since the cappable r.e. degrees form an ideal by Ambos-Spies, Jockusch, Shore, Soare [2]. But then 1 cannot be mapped to $\mathbf{0}'$, a contradiction. ■

Remarks. (i) The condition in the above theorem is not equivalent to requiring the lattice to have no complemented elements $\neq 0, 1$ (which was our first conjecture for the condition in the theorem). For example, the free distributive lattice with three generators has no complemented elements but all its join-irreducible elements are cappable.

(ii) We claim that our theorem easily extends to the countable case, namely, a countable distributive lattice is embeddable into the r.e. degrees preserving least and greatest element iff the lattice contains a prime filter of noncappable elements. (In the finite case, this prime filter is the principal filter generated by the distinguished join-irreducible noncappable element.)

Our theorem implies the solutions to two long-open questions:

Corollary 1. *The double diamond lattice can be embedded into the r.e. degrees preserving least and greatest element.*

Corollary 2. *There are two incomparable r.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such* $\mathbf{a} \cap \mathbf{b}$ *exists and exactly one of them is a cappable r.e. degree.*

Before giving the proofs of the corollaries, we state two more definitions:

Definition 2. Let $\mathcal{L} = \langle L, \leq, \vee, \wedge, 0, 1\rangle$ be a lattice with least and greatest element.

(i) $a \in L$ is an *atom* of $\mathcal{L}$ if $a \neq 0$ and there is no $b \in L$ with $0 < b < a$.

(ii) $a \in L$ is a *coatom* of $\mathcal{L}$ if $a \neq 1$ and there is no $b \in L$ with $a < b < 1$.

Proof of Corollary 1. Either coatom of the double diamond lattice is both join-irreducible and noncappable, thus the corollary follows by our theorem. ■

Proof of Corollary 2. Let a and b be the coatoms and c and d be the atoms of the double diamond lattice. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ be their images under the embedding in Corollary 1. Then $\mathbf{a}$ and $\mathbf{b}$ are incomparable, and their meet exists (and is equal to $\mathbf{c} \cup \mathbf{d}$). Now recall that the cappable r.e. degrees form a nontrivial ideal of the r.e. degrees, and that the noncappable r.e. degrees form a nontrivial filter of the r.e. degrees (both by Ambos-Spies, Jockusch, Shore, Soare [2]). If $\mathbf{a}$ and $\mathbf{b}$ were both cappable then so would be their join, $\mathbf{0}'$, a contradiction. If $\mathbf{a}$ and $\mathbf{b}$ were both noncappable then so would be their meet $\mathbf{a} \cap \mathbf{b} = \mathbf{c} \cup \mathbf{d}$, but it is also the join of two cappable degrees, again a contradiction. Thus exactly one of $\mathbf{a}$ and $\mathbf{b}$ is cappable. ■

Before proving the main theorem, we will review some easy lattice-theoretical facts and fix some notation.

3. Lattice theory: Notation and facts. From now on until the end of this paper, we fix a finite distributive lattice $\mathcal{L} = \langle L, \leq, \vee, \wedge, 0, 1\rangle$ and the following notation for $\mathcal{L}$: We let $J_{\mathcal{L}} = \{j_0, \cdots j_n\}$ be the set of join-irreducible elements of

$\mathcal{L}$. Let $\mathcal{J}_{\mathcal{L}}$ be the set of all downward closed subsets of $J_{\mathcal{L}}$. Call a pair of sets $J_1, J_2 \in \mathcal{J}_{\mathcal{L}}$ *incomparable* if $J_1 \not\subseteq J_2$ and $J_2 \not\subseteq J_1$. We denote by $\mathcal{J}^2_{\mathcal{L}}$ the set of all incomparable pairs of sets in $\mathcal{J}_{\mathcal{L}}$.

We can now state the following

Lattice-theoretical facts. *Let $\mathcal{L}$ be as above. Then:*

(1) *If $a, b \in L$, $j \in J_{\mathcal{L}}$, and $a \vee b \geq j$ then $a \geq j$ or $b \geq j$.*
(2) *Each $a \in L$ can be written uniquely as the join of a set $J \in \mathcal{J}_{\mathcal{L}}$. (Here, we let 0 be the join of $\emptyset$.)*
(3) *For each pair $a, b \in L - \{1\}$ with $a \vee b = 1$ there are nonempty sets $J_1, J_2 \subseteq J_{\mathcal{L}}$ such that $a \geq \bigvee J_1$, $b \geq \bigvee J_2$, and $J_1 \cup J_2 = J_{\mathcal{L}}$.*
(4) *If $J_1, J_2 \in \mathcal{J}_{\mathcal{L}}$ then $\bigvee J_1 \wedge \bigvee J_2 = \bigvee (J_1 \cap J_2)$.*

Proof. (1) If $a \vee b \geq j$ then $j = j \wedge (a \vee b) = (j \wedge a) \vee (j \wedge b)$ by distributivity and thus $j = j \wedge a$ or $j = j \wedge b$ by the join-irreducibility of j, showing $j \leq a$ or $j \leq b$.

(2) Set $J = \{j \in J_{\mathcal{L}} \mid j \leq a\}$. Then clearly $J \in \mathcal{J}_{\mathcal{L}}$, and $a = \bigvee J$ by the definition of join-irreducibility and the finiteness of $\mathcal{L}$. Now suppose $a = \bigvee J = \bigvee J'$ for two distinct sets $J, J' \in \mathcal{J}_{\mathcal{L}}$. By symmetry assume $j \in J - J'$ for some $j \in J_{\mathcal{L}}$. By $j \in J$, $j \leq a = \bigvee J'$, and so $j \leq j'$ for some $j' \in J'$ by (1), contradicting the downward-closedness of J'.

(3) Let $J_1 = \{j \in J_{\mathcal{L}} \mid j \leq a\}$ and $J_2 = \{j \in J_{\mathcal{L}} \mid j \leq b\}$. First, for any $j \in J_{\mathcal{L}}, j \leq 1 = a \vee b$, and thus $j \leq a$ or $j \leq b$ by Fact (1). Thus $J_1 \cup J_2 = J_{\mathcal{L}}$. Suppose $J_i = \emptyset$ for $i = 1$ or 2. (By symmetry, assume $i = 1$.) Then $J_2 = J_{\mathcal{L}}$ since $J_1 \cup J_2 = J_{\mathcal{L}}$. Thus $b \geq \bigvee J_2 = \bigvee J_{\mathcal{L}} = 1$, the last equality following by Fact (2). But this contradicts $b < 1$.

(4) Set $a = \bigvee J_1$ and $b = \bigvee J_2$. We will show that $j \in J_1 \cap J_2$ iff $j \leq a \wedge b$. First, $j \in J_1 \cap J_2$ implies $j \leq a, b$ and thus $j \leq a \wedge b$. Conversely, $j \leq a \wedge b$ implies $j \leq a = \bigvee J_1$ and $j \leq b = \bigvee J_2$. By Fact (1), there are $j_1 \in J_1$ and $j_2 \in J_2$ such that $j \leq j_1$ and $j \leq j_2$. By the downward closedness of J_1 and J_2, we have $j \in J_1$ and $j \in J_2$ and thus $j \in J_1 \cap J_2$ as desired. ■

We now start on the proof of the hard direction of our theorem.

4. The Requirements. Recall the notation from the previous section. We assume from now on that j_0 is a join-irreducible noncappable element of $\mathcal{L}$.

If 1 is join-irreducible then $L - \{1\}$ forms a finite distributive lattice, which, by Thomason [9] and Lerman (unpublished), is embeddable into the r.e. degrees preserving 0. This easily extends to the desired embedding of $\mathcal{L}$. We may thus from now on assume $1 \notin J_{\mathcal{L}}$.

For each $j \in J_{\mathcal{L}}$, we will build an r.e. set B_j. For any set $S \in \mathcal{J}_{\mathcal{L}}$, we set $A_S = B_{j_{i_0}} \oplus \cdots \oplus B_{j_{i_k}}$ where $S = \{j_{i_0}, \cdots, j_{i_k}\}$ and $0 \leq i_0 < \cdots < i_k \leq n$. (For $S = \emptyset$, we set $A_S = \emptyset$.) For each $a \in L - \{1\}$, we set $A_a = A_J$ for the unique $J \in \mathcal{J}_{\mathcal{L}}$ with $a = \bigvee J$ (using Fact (2) from the previous section). We set $A_1 = K$ for the complete r.e. set K. Then the map $a \mapsto \deg A_a$ will be the desired embedding.

We will also build one partial recursive (p.r.) functional Γ and many p.r. functionals Δ (to be distinguished later by subscripts). Our requirements will then be as follows:

For each x, we require

$$\mathcal{C}_x : K(x) = \Gamma(A_{J_{\mathcal{L}}}; x).$$

For each incomparable pair $(J_1, J_2) \in \mathcal{J}_{\mathcal{L}}^2$ and each p.r. functional Φ, we require

$$\mathcal{M}^{\Phi}_{J_1,J_2} : \Phi(A_{J_1}) = \Phi(A_{J_2}) \text{ total } \to \exists \Delta(\Delta(A_{J_1 \cap J_2}) = \Phi(A_{J_1})).$$

For each $j \in J_{\mathcal{L}}$ and each p.r. functional Ψ, we require

$$\mathcal{P}^{\Psi}_j : B_j \neq \Psi(A_{J_j})$$

where $J_j = \{j' \in J_{\mathcal{L}} \mid j \not\leq j'\}$.

We claim that the above requirements suffice to ensure that our embedding is as desired, namely, that we have for all $a, b, c \in L$,

$$a \leq b \to A_a \leq_{\mathrm{T}} A_b, \tag{1a}$$

$$a \not\leq b \to A_a \not\leq_{\mathrm{T}} A_b, \tag{1b}$$

$$a \vee b = c \to A_a \oplus A_b \equiv_{\mathrm{T}} A_c, \tag{2}$$

$$a \wedge b = c \to \deg A_a \cap \deg A_b = \deg A_c, \tag{3}$$

$$A_0 = \emptyset, \text{ and} \tag{4}$$

$$A_1 = K. \tag{5}$$

Here, (1a), (4), and (5) are trivial by the definition of A_a.

For (1b), use Fact (2) from the previous section to obtain some $j \in J_{\mathcal{L}}$ with $j \leq a$ and $j \not\leq b$ and thus $b \leq \bigvee J_j$. Now the requirements $\mathcal{P}^{\Psi}_j$ will ensure (1b).

In (2) and (3), we may assume a and b to be incomparable, otherwise (1a) will suffice to establish (2) and (3). Furthermore, (2) is trivial for $a = 1$ or $b = 1$ or $c < 1$ by the definition of A_c.

(2) in the case $a, b < 1$ and $c = 1$ is ensured by the requirements $\mathcal{C}_x$ as follows. Let J_1 and J_2 be as in Fact (3) from the previous section. Then

$$A_1 = K \leq_{\mathrm{T}} A_{J_{\mathcal{L}}} \leq_{\mathrm{T}} A_{J_1} \oplus A_{J_2} \leq_{\mathrm{T}} A_a \oplus A_b \leq_{\mathrm{T}} K = A_1,$$

using Fact (3) and the definition of A_a and A_b.

(3) in the case of incomparable a and b is ensured by the requirements $\mathcal{M}^{\Phi}_{J_1,J_2}$ by Posner's trick (see Soare [8]) and since if $a = \bigvee J_1$ and $b = \bigvee J_2$ for some sets $J_1, J_2 \in \mathcal{J}_{\mathcal{L}}$ then (J_1, J_2) must be an incomparable pair and $c = \bigvee(J_1 \cap J_2)$ by Fact (4) from the previous section.

We have thus shown that our requirements ensure (1a)-(5).

5. The strategies. We have three types of strategies, one for each type of requirement $\mathcal{C}_x$, $\mathcal{M}^{\Phi}_{J_1,J_2}$, and $\mathcal{P}^{\Psi}_j$, and two methods to coordinate these strategies, the "stages method" and the "block method".

The *stages method* simply consists in allowing numbers to enter at most one set, B_j say (and possibly B_{j_0} if $j \leq j_0$), at any given stage s. In that case, we call s a *j-stage*. (The stages method will be important in verifying the satisfaction of the $\mathcal{M}$-requirements (i.e. the infimum requirements).)

A strategy ε for $\mathcal{P}^{\Psi}_j$ uses diagonalization. It chooses a witness z for diagonalization, waits for a computation $\Psi(A_{J_j}; z) \downarrow = 0$, puts z into B_j, and restrains A_{J_j} (i.e. any set $B_{j'}$ with $j' \in J_j$) on numbers $\leq$ the use $\psi(z)$. This restraint is imposed on lower-priority $\mathcal{C}_x$-strategies by "$\mathcal{C}_x$-lifting" (as explained more closely below), namely, roughly speaking, by putting the use $\gamma(x)$ into B_j and forcing the new use $\gamma(x)$ to exceed ε's restraint.

A strategy β for $\mathcal{M}^{\Psi}_{J_1,J_2}$ will construct a partial recursive (p.r.) functional $\Delta_\beta(A_{J_0})$ (where $J_0 = J_1 \cap J_2$). When $\Phi(A_{J_1})$ and $\Phi(A_{J_2})$ seem to agree on a longer initial segment than ever before (i.e. at a so-called "expansionary" stage) then β extends the domain of $\Delta_\beta(A_{J_0})$ to that initial segment. If $J_0 \neq \emptyset$ then whenever $\Delta_\beta(A_{J_0}; y)$ is defined but agrees with neither $\Phi(A_{J_1}; y)$ nor $\Phi(A_{J_2}; y)$ then β will correct $\Delta_\beta(A_{J_0}; y)$ by putting a number $\leq$ the use $\delta_\beta(y)$ into A_{J_0}, i.e. into a set B_j for some $j \in J_0$. The rest of the construction will determine which j is chosen by the stages method mentioned above and the block method to be described below. The hardest part of the proof will be to show that if $J_0 = \emptyset$ (and thus $\Delta_\beta(A_{J_0})$ cannot be corrected since $A_{J_0} = \emptyset$) then we will never want to correct $\Delta_\beta(A_{J_0})$. (Note that if $J_0 = \emptyset$ then $\mathcal{M}^{\Phi}_{J_1,J_2}$ is a minimal pair requirement. Essential ingredients of the proof here will be the block method and the fact that $j_0 \notin J_1 \cup J_2$ since j_0 is noncappable in $\mathcal{L}$.)

The way we handle the $\mathcal{C}_x$-requirements is new and at the heart of our proof. First of all, despite the global nature of the "overall" requirement $K = \Gamma(A_{J_\mathcal{L}})$, it is more convenient to split this up into the "sub"requirements $\mathcal{C}_x$, which will provide an easy way to assign priority ranking with respect to the other strategies. Satisfying $\mathcal{C}_x$ in the context of the other requirements consists in two parts, "$\mathcal{C}_x$-lifting" and "$\mathcal{C}_x$-correction". Since a whole level of the tree T of strategies (i.e. all strategies of some fixed length) will work on $\Gamma(A_{J_\mathcal{L}}; x)$ we have to decide which α will control $\Gamma(A_{J_\mathcal{L}}; x)$. We agree that control always stays with the leftmost $\mathcal{C}_x$-strategy, α_x say, that has been eligible to act so far and has not been initialized since. Whenever control passes from a $\mathcal{C}_x$-strategy α to a $\mathcal{C}_x$-strategy $\alpha' <_L \alpha$ then we increase the use $\gamma(x)$ by putting the old use $\gamma(x)$ into $A_{J_\mathcal{L}}$ (i.e. into some set B_j). We call this *$\mathcal{C}_x$-lifting.*

This happens, for example, if a higher-priority $\mathcal{P}$-strategy ε diagonalizes and helps in enforcing ε's restraint as explained above.

$\mathcal{C}_x$-correction applies when x enters K at a stage s, say. Say α_x controls $\Gamma(A_{J_\mathcal{L}}; x)$ at that stage. Typically, α_x will appear to be to the left of the true path at that stage. In that case, we will correct $\Gamma(A_{J_\mathcal{L}}; x)$ by putting the use $\gamma(x)$ into B_{j_0}. (Otherwise, i.e. if α_x does not appear to be to the left of the true path, we correct via B_j where j is determined by the stages method, which will cause no problems.)

Let us analyze more closely the case when α_x appears to be to the left of the true path. Consider the $\mathcal{M}$-strategies β with $\beta\hat{\ }\langle\infty\rangle \subseteq \alpha_x$, i.e. the $\mathcal{M}$-strategies β of which α_x assumes the infinite outcome. Since α_x appears to be to the left of the true path, for some of these β, s is not an expansionary stage, and so $\Delta_\beta(A_{J_0})$ may disagree with $\Phi(A_{J_1})$ or $\Phi(A_{J_2})$, say with the former. The $\mathcal{C}_x$-correction may destroy the other side, $\Phi(A_{J_2})$, say, and then β will need to correct $\Delta_\beta(A_{J_0})$.

It is here that we need the block method. First of all, if $\Phi(A_{J_1})$ or $\Phi(A_{J_2})$ is destroyed for β by $\mathcal{C}_x$-correction then β cannot be a minimal pair strategy (i.e. an $\mathcal{M}$-strategy with $J_0 = \emptyset$ and a $\Delta_\beta(A_{J_0})$ that cannot be corrected) since j_0 is noncappable in $\mathcal{L}$ and thus $j_0 \notin J_1 \cup J_2$ in that case. It is still conceivable that a minimal pair strategy is injured indirectly by the correction of other injured $\mathcal{M}$-strategies.

We prevent this from happening by the *block method.* We arrange the $\mathcal{M}$-strategies β with $\beta\hat{\ }\langle\infty\rangle \subseteq \alpha_x$ into *blocks* $\mathcal{B}$, each block $\mathcal{B}$ containing those β that had their last expansionary stage at the same stage $s_\mathcal{B}$, say, $\leq$ the current stage. Then two $\mathcal{M}$-strategies β and β' in the same block will not interfere with each other since they share the last expansionary stage and then were both injured "on the same side", so they are able to cooperate. We will be able to show that by initialization, any two $\mathcal{M}$-strategies β and β' in distinct blocks cannot interfere with each other.

We finally remark that in order to make $\mathcal{C}_x$-lifting work properly we will ensure that any strategy that acts at a stage s_1 and is initialized at a stage $s_2 > s_1$ cannot act at a stage $s_3 \geq s_2$.

After these intuitive explanations, we will now describe the construction formally.

6. The Construction. Let $\Lambda = \{\infty\} \cup \omega^*$ be the *set of outcomes* where ∞ is a symbol (for the infinite outcome) and ω^* is the set ω with the reverse order. We order Λ as ω^* with ∞ leftmost. Our construction uses a *tree of strategies*

$$T = \{\xi \in \Lambda^{<\omega} \mid \forall i(\xi(i) \downarrow <_\Lambda 1 \rightarrow i \equiv 1 \mod 3)\} \tag{6}$$

We effectively ω-order the $\mathcal{M}^{\Phi}_{J_1,J_2}$- and $\mathcal{P}^{\Psi}_{j}$-requirements as $\{\mathcal{M}_i\}_{i\in\omega}$ and $\{\mathcal{P}_i\}_{i\in\omega}$, respectively. We *assign* the $\mathcal{C}_x$-requirements to all $\xi \in T$ with $|\xi| = 3x$, the $\mathcal{M}_i$-requirements to all $\xi \in T$ with $|\xi| = 3i + 1$, and the $\mathcal{P}_i$-requirements to all $\xi \in T$ with $|\xi| = 3i + 2$, for all $x, i \in \omega$, respectively. If requirement $\mathcal{R}$ is assigned to $\xi \in T$, we say ξ *works for* $\mathcal{R}$, and we call ξ an *$\mathcal{R}$-strategy.* (Note that, by (6), the only possible outcomes of a $\mathcal{C}$- or $\mathcal{P}$-strategy are 1 and 0.)

A strategy ξ is initialized by making all its parameters undefined; if ξ is an $\mathcal{M}$-strategy, by also making Δ_ξ totally undefined; and, if $\xi = \alpha_x$ for some x, by also making α_x undefined.

A parameter is *defined big* by setting it equal to a number greater than any number mentioned so far in the construction.

For the $\mathcal{C}_x$-strategies, we fix a recursive enumeration $\{K_s\}_{s\in\omega}$ of K such that $|K_s| = s$ (and thus $|K_{s+1} - K_s| = 1$) for all s.

For the $\mathcal{M}^{\Phi}_{J_1,J_2}$-strategies, we define the *length of agreement*

$$\ell = \max\{y \mid \forall y' < y(\Phi(A_{J_1}; y') \downarrow = \Phi(A_{J_2}; y') \downarrow)\}. \tag{7}$$

Our construction proceeds in *stages.* At stage 0, we initialize all strategies $\xi \in T$, set $B_j = \emptyset$ for all $j \in J_{\mathcal{L}}$, let Γ be totally undefined, call 0 a *β-expansionary stage* for all $\mathcal{M}$-strategies $\beta \in T$, let α_x (the $\mathcal{C}_x$ -strategy currently *controlling* $\Gamma(A_{J_{\mathcal{L}}}; x)$) be undefined for all x, say there are currently no *blocks* of $\mathcal{M}$-strategies, *declare* 0 *a j_0-stage*, and set $\delta_0 = \emptyset$.

At a stage $s + 1$, we first fix $x_0 \in K_{s+1} - K_s$.

At each substage $t \leq s$ of stage $s + 1$, a strategy $\xi \in T$ of length t is "eligible to act" and, at the end of the substage t, determines the strategy $\xi' \supset \xi$ eligible to act next (unless $t = s$).

At substage t of stage $s+1$, we distinguish three cases depending upon the type of the strategy eligible to act at this substage.

Case 1: A $\mathcal{C}_x$-strategy α is eligible to act at substage t. We say α *requires $\mathcal{C}_x$-lifting* if $\alpha_x >_L \alpha$ or α_x is undefined. We say α *requires $\mathcal{C}_x$-correction* if $\Gamma(A_{J_{\mathcal{L}}}; x) \downarrow \neq K_{s+1}(x)$ (and thus $x = x_0$). If α requires $\mathcal{C}_x$-lifting or $\mathcal{C}_x$-correction, α puts the (current or most recent) use $\gamma(x)$ into B_j (if $s + 1$ has already been declared a j-stage and no $\mathcal{M}$-strategy $\beta \subset \alpha$ performs $\mathcal{M}$-correction at stage $s + 1$), or into B_{j_0} (otherwise, then *declaring* $s + 1$ *a j_0-stage* if $s + 1$ has not yet been declared a

j-stage for any $j \in J_{\mathcal{L}}$); we say α *performs* $\mathcal{C}_x$*-lifting* or $\mathcal{C}_x$*-correction*, respectively. If α performs $\mathcal{C}_x$-lifting then we set $\alpha_x = \alpha$. In any case, if now $\Gamma(A_{J_{\mathcal{L}}}; x)$ is undefined then α sets $\Gamma(A_{J_{\mathcal{L}}}; x) = K_{s+1}(x)$ with previous use $\gamma(x)$ (if $\Gamma(A_{J_{\mathcal{L}}}; x)$ was previously defined and no $\gamma(x')$ (for any $x' \leq x$) has entered any B_j since $\Gamma(A_{J_{\mathcal{L}}}; x)$ was last defined) or with big use $\gamma(x)$ (otherwise). Finally, α ends the substage by letting $\alpha\hat{\ }\langle K_{s+1}(x)\rangle$ be *eligible to act next.*

Case 2: An $\mathcal{M}^{\Phi}_{J_1,J_2}$-strategy β is eligible to act at substage t. Set $J_0 = J_1 \cap J_2$. We say β *should allow* $\mathcal{C}_{x_0}$*-correction* if $\beta\hat{\ }\langle\infty\rangle \subseteq \alpha_{x_0}$ and $\Gamma(A_{J_{\mathcal{L}}}; x_0) \downarrow= 0$. We say β *requires* $\mathcal{M}$*-correction* if there is some y such that $\Delta_\beta(A_{J_0}; y)$ is defined but there are stages $s_1, s_2 \in [s_0, s+1]$ (where s_0 is the greatest β-expansionary stage $\leq s$) such that $\Delta_\beta(A_{J_0}; y) \neq \Phi(A_{J_1}; y)[s_1]$ *and* $\Delta_\beta(A_{J_0}; y) \neq \Phi(A_{J_2}; y)[s_2]$. We say β *requires* $\mathcal{M}$*-definition* if $\ell[s+1] > \ell[s']$ for any stage $s' \leq s$ at which β was eligible to act.

We distinguish three subcases (if none applies go to the end of substage t):

Subcase 2a: β should allow $\mathcal{C}_{x_0}$-correction, but either β does not require $\mathcal{M}$-definition or β requires $\mathcal{M}$-correction. Then β will let α_{x_0} *perform* $\mathcal{C}_{x_0}$*-correction* as follows. First α_{x_0} puts $\gamma(x_0)$ into B_{j_0}. Let

$$\mathcal{B}_{s+1} = \{\beta' \ \mathcal{M}\text{-strategy} \mid \beta'\hat{\ }\langle\infty\rangle \subseteq \alpha_{x_0}\}, \text{ and} \tag{8}$$

$$S_{s+1} = \{s' \leq s+1 \mid \exists \beta' \in \mathcal{B}_{s+1} \ (s' \text{ is the greatest } \beta'\text{-expansionary stage} \leq s+1)\}. \tag{9}$$

For each $s' \in S_{s+1}$, *create a block*

$$\mathcal{B} = \{\beta' \in \mathcal{B}_{s+1} \mid s' \text{ is the greatest } \beta'\text{-expansionary stage} \leq s+1)\}. \tag{10}$$

For each block $\mathcal{B}$, let $s_{\mathcal{B}}$ be the corresponding s' and fix $j_{\mathcal{B}} \in J_{\mathcal{L}}$ with $j_{\mathcal{B}} \leq j_0, j$ and $j_{\mathcal{B}} = j_i$ for the least possible i (where s' was a j-stage). ($j_{\mathcal{B}}$ must exist since j_0 is noncappable in $\mathcal{L}$.) α_{x_0} *declares* $s+1$ *a* j_0*-stage.* (We will show later (in Lemma 3) that $s+1$ cannot have been declared a j-stage before for any j.) α_{x_0} initializes all strategies $\eta > \alpha_{x_0}$ and discards all blocks containing only initialized strategies. We also call $s+1$ a β'*-correction stage* for all $\beta' \in \mathcal{B}_{s+1}$ with $\beta' \supseteq \beta$.

Subcase 2b: β requires $\mathcal{M}$-correction but should not allow $\mathcal{C}_{x_0}$-correction. Then, as we will show later (in Lemma 6), β must currently be in one (or more) block(s) $\mathcal{B}$ with a unique parameter $j_{\mathcal{B}}$. Let β require $\mathcal{M}$-correction via some (least) y. Fix the greatest $y_0 \leq y$ such that for all $y' < y_0$, $\Delta_\beta(A_{J_0}; y') = \Phi(A_{J_i}; y') \downarrow$ and $\varphi(A_{J_i}; y) < \delta_\beta(y_0)$ for some $i \in \{1, 2\}$. β puts $\delta_\beta(y_0)$ into B_{j_0} (if $j_0 \in J_0$, *declaring* $s+1$ *a* j_0*-stage*) or into $B_{j_{\mathcal{B}}}$ (otherwise, *declaring* $s+1$ *a* $j_{\mathcal{B}}$*-stage*). (Note that correcting $\Delta_\beta(A_{J_0}; y)$ will inductively require correcting $\Delta_\beta(A_{J_0}; y_0)$ also. We will show later (in Lemma 3) that $s+1$ cannot have been declared a j-stage before for any $j \in J_{\mathcal{L}}$). We say β *performs* $\mathcal{M}$*-correction.* We call $s+1$ a β*-correction stage.*

Subcase 2c: β requires $\mathcal{M}$-definition but does not require $\mathcal{M}$-correction. Then, for all $y < \ell[s+1]$ for which $\Delta_\beta(A_{J_0}; y)$ is now undefined, β sets $\Delta_\beta(A_{J_0}; y) = \Phi(A_{J_1}; y)$ with previous use $\delta_\beta(y)$ (if $\Delta_\beta(A_{J_0}; y)$ was previously defined and no $\delta_\beta(y')$ (for any $y' \leq y$) has entered any B_j since $\Delta_\beta(A_{J_0}; y)$ was last defined) or with big use $\delta_\beta(y)$ (otherwise). We remove β from any block $\mathcal{B}$ that it currently is in, if any. We call $s+1$ a β*-expansionary stage.*

At the end of substage t, β ends the substage by letting $\beta\hat{}\langle\infty\rangle$ be *eligible to act next* (if $s+1$ is a β-expansionary stage); or by letting $\beta\hat{}\langle s'\rangle$ be *eligible to act next* (otherwise, where s' is the greatest β-expansionary or β-correction stage $\leq s+1$).

Case 3: A $\mathcal{P}_j^\Psi$-strategy ε is eligible to act at substage t. Set $J_j = \{j' \in J_\mathcal{L} \mid j \not\leq j'\}$. If ε currently has no witness z then it picks the *witness* z big. If $\Psi(A_{J_j}; z) \downarrow= 0 = B_j(z)$ then put z into B_j and *declare* $s+1$ *a* j*-stage.* (In this case, we say ε *diagonalizes.* We will show later (in Lemma 3 (i)) that $s+1$ cannot have been declared a j'-stage before for any j'.) ε ends the substage by letting $\varepsilon\hat{}\langle B_j(z)\rangle$ be *eligible to act next.*

This ends the description of the action at substages. At the end of stage $s+1$, i.e. after substage s of stage $s+1$, we perform the following additional action: Let δ_{s+1}, the *approximation to the true path*, be the strategy eligible to act at substage s of stage $s+1$. We initialize all strategies $\xi > \delta_{s+1}$. Next, we *declare* $s+1$ *a* j_0*-stage* if it has not yet been declared a j-stage for any j. Furthermore, we discard any blocks containing strategies initialized at stage $s+1$. Finally, we let all $\mathcal{M}$-strategies β with $\beta\hat{}\langle\infty\rangle \subseteq \delta_{s+1}$ that now require $\mathcal{M}$-correction (as defined in Case 2) perform $\mathcal{M}$-correction (as defined in Subcase 2b) via B_j (if $j \in J_0^\beta$ where $s+1$ is a j-stage) or via B_{j_0} (if $j \leq j_0$ and $j \notin J_0^\beta$) until no such β requires $\mathcal{M}$-correction. (We will show later (in Lemma 6) that any such β *can* perform $\mathcal{M}$-correction this way.)

This completes the description of the construction.

7. The Verification.

We begin by defining the true path of the construction:

Definition 3. The *true path* $f \in [T]$ is defined inductively by

$$f(n) = \mu i \in \Lambda\, \exists^\infty s((f \restriction n)\hat{}\langle i\rangle \subseteq \delta_s). \tag{11}$$

We first show that this is a good definition:

Lemma 1 (True Path Lemma). *(i) If an $\mathcal{M}$-strategy β has infinitely many β-correction stages then it has infinitely many β-expansionary stages.*

(ii) The true path is well-defined.

Proof. (i) Suppose β has only finitely many β-expansionary stages, say s_0 is the greatest such. Then, at s_0, there are only finitely many $\mathcal{C}_x$-strategies of the form $\alpha_x \supseteq \beta\hat{}\langle\infty\rangle$, and no new $\alpha_x \supseteq \beta\hat{}\langle\infty\rangle$ can be added after s_0; likewise, at s_0, there are only finitely many y such that $\Delta_\beta(A_{J_0}; y)\downarrow$, and no new definitions of $\Delta_\beta(A_{J_0}; y)$ for any y can be made after s_0. But for each β-correction stage $> s_0$, x enters K for some such α_x, or $\Delta_\beta(A_{J_0}; y)$ is destroyed for some such y. Thus there can be only finitely many β-correction stages $> s_0$ as desired.

(ii) We note that $|\delta_{s+1}| = s$ for all s. We proceed by induction on n in (11). By the definition of T in (6), i trivially exists for $n \equiv 0$ or $2 \mod 3$, i.e. for $\mathcal{C}$- or $\mathcal{P}$-strategies $f \restriction n$. For $n \equiv 1 \mod 3$, i.e. for an $\mathcal{M}$-strategy $\beta = f \restriction n$, we conclude by (i) that either β has a greatest β-expansionary or β-correction stage s_0, and thus $\beta\hat{}\langle s_0\rangle \subseteq f$; or β has infinitely many β-expansionary stages, and thus $\beta\hat{}\langle\infty\rangle \subseteq f$. ∎

We next turn to initialization.

Lemma 2 (Initialization Lemma). *(i) Every strategy $\xi \subset f$ is initialized at most finitely often.*

(ii) If a strategy ξ is eligible to act at a stage s_1 and is then initialized at a stage $s_2 > s_1$, then it is not eligible to act at any stage $s_3 \geq s_2$.

Proof. (i) We proceed by induction on $|\xi|$. (i) is trivial for $\emptyset$ since $\emptyset$ is never initializedafter stage 0. Fix $\xi \subset f$ with $\xi \neq \emptyset$ and assume (i) for $\xi^- = \xi \restriction (|\xi| - 1)$. By the definition of f in (11), ξ cannot be initialized infinitely often at the end of a stage. Thus, by induction hypothesis, if ξ is initialized infinitely often then ξ^- is an $\mathcal{M}$-strategy and $\xi \neq \xi^- \hat{\ } \langle\infty\rangle$. But then there are infinitely many ξ^--correction stages, contradicting Lemma 1(i) and $\xi^- \hat{\ } \langle\infty\rangle \not\subset f$.

(ii) We again proceed by induction on $|\xi|$. As in (i), (ii) is trivial for $\emptyset$. So suppose ξ is initialized at s_2, and let $\eta \subset \xi$ be the longest substring not initialized at that time (using that $\emptyset$ is never initialized). We distinguish three cases by the type of strategy of η. If η is a $\mathcal{C}_x$-strategy then $\eta \hat{\ } \langle 0\rangle \subseteq \xi$ and at a stage $s \in (s_1, s_2]$, x must have entered K, so ξ cannot be eligible to act at a stage $\geq s_2$. If η is a $\mathcal{P}_j^\Psi$-strategy then $\eta \hat{\ } \langle 0\rangle \subseteq \xi$ and η's witness z must have entered B_j at a stage $s \in (s_1, s_2]$, so again ξ cannot be eligible to act at a stage $\geq s_2$. Finally, if η is an $\mathcal{M}$-strategy then, by the way initialization is arranged, $\eta \hat{\ } \langle\infty\rangle \not\subseteq \xi$, and in fact $\eta \hat{\ } \langle s'\rangle \subseteq \xi$ for some $s' \leq s_1$. But by our assumptions on η and s_2, s_2 must be an η-expansionary or η-correction stage, so ξ cannot be eligible to act at a stage $\geq s_2$. ■

We now show that the stages method works properly.

Lemma 3 (Stages Lemma). *Each stage is declared a j-stage (for some $j \in J_{\mathcal{L}}$) exactly once, and no number enters any set $B_{j'}$ (for $j' \neq j$) at that stage (except possibly B_{j_0} if $j \leq j_0$).*

Proof. The lemma is trivial for stage 0, and by the action at the end of a stage, each stage is declared a j-stage (for some $j \in J_{\mathcal{L}}$) *at least* once.

Now fix a stage $s+1$ and assume first that $s+1$ is declared a j-stage (for some $j \in J_{\mathcal{L}}$) only at the end of stage $s+1$, i.e. after substage s. Then no numbers entered any set $B_{j'}$ (for any $j' \in J_{\mathcal{L}}$) at any substage (or else $s+1$ would have been declared a j'-stage then), and no numbers enter any set $B_{j'}$ since any $\mathcal{M}$-strategy $\beta \subseteq \delta_{s+1}$ that would perform $\mathcal{M}$-correction now would have done so at substage $|\beta|$ already.

Now assume that $s+1$ is (first) declared a j-stage (for some $j \in J_{\mathcal{L}}$) at substage t by some $\xi \in T$. Then, by the construction, no numbers can enter any set $B_{j'}$ (for any $j' \in J_{\mathcal{L}}$) at stage $s+1$ before substage t. We distinguish cases depending on the type of strategy of ξ.

In all the cases below, we will use the fact that if $\xi' \subseteq \delta_{s+1}$ has not been eligible to act at a stage $\leq s$ but is eligible to act at a substage $> t$ of stage $s+1$ then ξ' will not declare $s+1$ a j'-stage (for any $j' \in J_{\mathcal{L}}$) and will not put numbers into any set $B_{j'}$ (for any $j' \neq j$ except possibly $j' = j_0$ if $j \leq j_0$). This is clear if ξ' is an $\mathcal{M}$- or a $\mathcal{P}$-strategy. If ξ' is a $\mathcal{C}_x$-strategy then ξ' cannot act via Case 2a of the construction (or else $\xi' = \alpha_x$ was eligible to act before). If ξ' is a $\mathcal{C}_x$-strategy and acts via Case 1 then it will at worst put $\gamma(x)$ into B_j (or B_{j_0}) since ξ already declared $s+1$ a j-stage.

Now suppose first that ξ performs $\mathcal{C}_x$-lifting. Then no $\xi' \supseteq \xi$ has ever been eligible to act before.

If ξ performs $\mathcal{C}_x$-correction via Case 1 of the construction then x just entered K, so $\xi \hat{\ } \langle 1\rangle \subseteq \delta_{s+1}$ and no $\xi' \supseteq \xi \hat{\ } \langle 1\rangle$ has ever been eligible to act before.

If ξ performs $\mathcal{C}_x$-correction via Case 2a of the construction for an $\mathcal{M}$-strategy $\beta \subset \xi$ then $\beta\hat{}\langle s+1\rangle \subseteq \delta_{s+1}$ and no $\xi' \supseteq \beta\hat{}\langle s+1\rangle$ has ever been eligible to act before.

If ξ is an $\mathcal{M}$-strategy (and declares $s+1$ a j-stage at substage t) then it must do so for $\mathcal{M}$-correction, and thus $\xi\hat{}\langle s+1\rangle \subseteq \delta_{s+1}$ and no $\xi' \supseteq \xi\hat{}\langle s+1\rangle$ has ever been eligible to act before.

Finally, if ξ is a $\mathcal{P}$-strategy (and declares $s+1$ a j-stage at substage t) then it must do so for diagonalization and put its witness z into B_j at this time, thus again $\xi\hat{}\langle 1\rangle \subseteq \delta_{s+1}$ and no $\xi' \supseteq \xi\hat{}\langle 1\rangle$ has ever been eligible to act before. ∎

We can now verify the satisfaction of our requirements.

Lemma 4 ($\mathcal{C}_x$-Satisfaction Lemma). $\Gamma(A_{J_\mathcal{L}}) = K$.

Proof. Fix x, the $\mathcal{C}_x$-strategy $\alpha \subset f$, and a stage s_0 at which α is eligible to act. Then at any stage $s \geq s_0$, $\alpha \leq \delta_s$, thus α_x is a fixed strategy $\leq \alpha$, and so no $\mathcal{C}_x$-strategy will perform $\mathcal{C}_x$-lifting after stage s_0. Now $\mathcal{C}_x$-correction (which occurs at most once) and $\mathcal{C}_x$-lifting are the only ways to increase the use $\gamma(x)$, and so $\gamma(x)$ settles down eventually. Since α will define $\Gamma(A_{J_\mathcal{L}}; x)$ whenever it is undefined, $\Gamma(A_{J_\mathcal{L}}; x)$ will eventually be defined forever.

If $x \notin K$ then trivially $\Gamma(A_{J_\mathcal{L}}; x) = K(x)$, so suppose x enters K at a stage $s+1$, say. Let $\overline{\alpha} = \alpha_x[s]$. If $3x > s$ then again trivially $\Gamma(A_{J_\mathcal{L}}; x) = K(x)$ since $\Gamma(A_{J_\mathcal{L}}; x)$ will never be defined before stage $s+1$. So suppose $3x \leq s$. If $\overline{\alpha} \not<_L \delta_{s+1}$ then $\delta_{s+1} \upharpoonright 3x = \alpha_x[s+1]$ will perform $\mathcal{C}_x$-correction (and also $\mathcal{C}_x$-lifting, if $\alpha_x[s+1] <_L \overline{\alpha}$), thus $\Gamma(A_{J_\mathcal{L}}; x) = K(x)$. Finally, if $\overline{\alpha} <_L \delta_{s+1}$ then let β be the longest common substring of $\overline{\alpha}$ and δ_{s+1}. Since α_x was eligible to act at a stage $\leq s$, necessarily β is an $\mathcal{M}$-strategy and $\beta\hat{}\langle\infty\rangle \subseteq \overline{\alpha}$. But then β will allow $\overline{\alpha}$ to perform $\mathcal{C}_x$-correction, and so again $\Gamma(A_{J_\mathcal{L}}; x) = K(x)$. ∎

Lemma 5 ($\mathcal{P}$-Satisfaction Lemma). *For all $j \in J_\mathcal{L}$ and all p.r. functionals Ψ, $B_j \neq \Psi(A_{J_j})$ (where $J_j = \{j' \in J_\mathcal{L} \mid j \not\leq j'\}$).*

Proof. Let $\varepsilon \subset f$ be a $\mathcal{P}_j^\Psi$-strategy and z its witness (which never changes by Lemma 2 (ii)). If $z \notin B_j$ then, by our construction, not $\Psi(A_{J_j}; z) \downarrow= 0$ as desired.

So suppose ε puts z into B_j at a stage s_0. Then $\Psi(A_{J_j}; z)[s_0] \downarrow= 0$ and ε declares s_0 a j-stage. Thus at no substage $> |\varepsilon|$ of stage s_0 will any strategy ξ put a number into A_{J_j} (i.e. into any set $B_{j'}$ for $j' \not\geq j$). We will show that, at any stage $> s_0$, no strategy ξ will put a number $\leq s_0$ into any set $B_{j'}$ (for any $j' \in J_\mathcal{L}$). This will establish $\Psi(A_{J_j}; z) \downarrow= 0$ as desired. So, for the sake of a contradiction, suppose this fails via some strategy $\xi \in T$ (acting first such). We distinguish cases for ξ.

If $\xi <_L \varepsilon$, or $\xi \subset \varepsilon$ but $\xi\hat{}\langle\infty\rangle \not\subseteq \varepsilon$, then ε would be initialized after stage s_0, contradicting Lemma 2 (ii).

If $\xi >_L \varepsilon$ then ξ will be initialized at stage s_0 and thus, from now on, not have any parameter $\leq s_0$. (This includes $\mathcal{C}_x$-strategies $\xi >_L \varepsilon$ by $\mathcal{C}_x$-lifting.)

If $\xi \supseteq \varepsilon\hat{}\langle 0\rangle$ then ξ will never be eligible to act at a stage $\geq s_0$ since $z \in B_j[s_0]$. Furthermore, since $|\delta_{s_0}| > |\delta_s|$ for any $s < s_0$, we will have $\alpha_x[s] \not\supseteq \varepsilon\hat{}\langle 0\rangle$ for all x and all $s \geq s_0$, and so $\mathcal{C}_x$-correction via a strategy $\xi \supseteq \varepsilon\hat{}\langle 0\rangle$ is impossible after stage s_0.

If $\xi \supseteq \varepsilon\hat{}\langle 1\rangle$ then ξ will not have been eligible to act before stage s_0 (since $x \notin B_j[s_0 - 1]$), and thus ξ will not have any parameter $\leq s_0$ unless it is a $\mathcal{C}_x$-strategy. But these $\mathcal{C}_x$-strategies $\xi \supseteq \varepsilon\hat{}\langle 1\rangle$ will perform $\mathcal{C}_x$-lifting via B_j at stage

s_0 unless $\alpha_x[s_0] <_L \varepsilon$, and so $\mathcal{C}_x$-correction via a strategy $\xi \supseteq \varepsilon\hat{\ }\langle 0\rangle$ after stage s_0 will not cause a number $\leq s_0$ to enter any set B'_j.

The only remaining case is that ξ is an $\mathcal{M}$-strategy, $\xi\hat{\ }\langle\infty\rangle \subseteq \varepsilon$, and that ξ might perform $\mathcal{M}$-correction to correct computations $\Delta_\xi(A_{J_0}; y)$ that were defined before stage s_0. But s_0 is a ξ-expansionary stage (since $\xi\hat{\ }\langle\infty\rangle \subseteq \varepsilon$), and when ξ is eligible to act at stage s_0, both $\Phi(A_{J_1}; y)$ and $\Phi(A_{J_2}; y)$ agree with $\Delta_\xi(A_{J_0}; y)$ for all the above y with uses $\varphi(A_{J_1}; y)$, $\varphi(A_{J_2}; y) < s_0$. If both $\Phi(A_{J_1}; y)$ and $\Phi(A_{J_2}; y)$ (for some such y) are destroyed later at stage s_0 then $j \in J_1$, J_2 and thus $j \in J_0$ so ξ will correct $\Delta_\xi(A_{J_0}; y)$ via B_j at the end of stage s_0. If one of $\Phi(A_{J_1}; y)$ and $\Phi(A_{J_2}; y)$ (for any such y) is not destroyed by the end of stage s_0 then it never will be destroyed by induction on this proof, and thus ξ will never want to correct $\Delta(A_{J_0}; y)$ for this y.

This establishes the lemma. ■

We now turn to the $\mathcal{M}$-requirements where the verification is more difficult. We begin with a lemma on the block method.

Lemma 6 (Block Lemma). *(i) If an $\mathcal{M}$-strategy $\beta \in T$ is in two blocks $\mathcal{B}$ and $\mathcal{B}'$ at the same time then $j_{\mathcal{B}} = j_{\mathcal{B}'}$ and $s_{\mathcal{B}} = s_{\mathcal{B}'}$.*

(ii) If an $\mathcal{M}^{\Phi}_{J_1,J_2}$-strategy $\beta \in T$ requires $\mathcal{M}$-correction (as defined in Case 2 of the construction) at a j-stage s then it will either perform $\mathcal{M}$-correction via B_j (or possibly B_{j_0} if $j \leq j_0$) at the end of stage s, or it is in a block $\mathcal{B}$ and will be able to perform $\mathcal{M}$-correction via $B_{j_{\mathcal{B}}}$.

Proof. (i) By Lemma 3, it suffices to show $s_{\mathcal{B}} = s_{\mathcal{B}'}$ since $s_{\mathcal{B}}$ is a $j_{\mathcal{B}}$-stage and $s_{\mathcal{B}'}$ is a $j_{\mathcal{B}'}$-stage. By the construction, both $s_{\mathcal{B}}$ and $s_{\mathcal{B}'}$ are β-expansionary stages, so suppose for the sake of a contradiction that $s_{\mathcal{B}} \neq s_{\mathcal{B}'}$, say $s_{\mathcal{B}} < s_{\mathcal{B}'}$. Then β is removed from $\mathcal{B}$ at a β-expansionary stage $\leq s_{\mathcal{B}'}$ (at substage $|\beta|$), and $\mathcal{B}'$ is created at a stage $\geq s'_{\mathcal{B}}$, and if at stage $s_{\mathcal{B}'}$ then at a substage $> |\beta|$. Thus β cannot be in $\mathcal{B}$ and $\mathcal{B}'$ at the same time.

(ii) Suppose β requires $\mathcal{M}$-correction at a (least) substage t of a stage s. We may assume that β did not require (this) $\mathcal{M}$-correction at stage $s-1$ by induction on s and since β can be removed from a block only by initialization or at a β-expansionary stage. Let s_0 be the greatest β-expansionary stage $\leq s$. We distinguish two cases:

Case 1: $s_0 = s$. Then t must be greater than $|\beta|$ since otherwise β would have a β-correction stage at stage s. Suppose s is a j-stage. Since s is β-expansionary, Lemma 3 implies $j \in J_1, J_2$ (or $j_0 \in J_1, J_2$ and $j \leq j_0$) and thus j (or j_0) $\in J_0 = J_1 \cap J_2$. Thus β will perform $\mathcal{M}$-correction via B_j (or B_{j_0}) at the end of stage s.

Case 2: $s_0 < s$. Since β has not been initialized since stage s_0, we have $\beta\hat{\ }\langle\infty\rangle <_L \delta_{s'}$ for all $s' \in (s_0, s]$. Furthermore, by $\mathcal{M}$-correction at the end of stage s_0, we have

$$\begin{aligned} &\forall \mathcal{M}\text{-strategies } \beta'\ \forall y(\beta'\hat{\ }\langle\infty\rangle \subseteq \beta\hat{\ }\langle\infty\rangle \wedge \Delta_{\beta'}(A_{J_0^{\beta'}}; y)[s_0]\downarrow \rightarrow \\ &\quad \exists i \in \{1,2\}(\Delta_{\beta'}(A_{J_0^{\beta'}}; y)[s_0] = \Phi_{\beta'}(A_{J_i^{\beta'}}; y)[s_0]\downarrow \wedge \\ &\qquad\qquad \varphi_{\beta'}(A_{J_i^{\beta'}}; y)[s_0] < s_0)). \end{aligned} \tag{12}$$

We will first show that if β is not in a block at substage t of stage s then no strategy $\xi \in T$ can put any number $\leq s_0$ into any set B_j (for any $j \in J_{\mathcal{L}}$) after stage s_0

until substage t of stage s, contradicting β requiring $\mathcal{M}$-correction at substage t of stage s. For the sake of a contradiction, suppose ξ exists (and is first acting such). By (12), ξ cannot be an $\mathcal{M}$-strategy with $\xi\hat{\ }\langle\infty\rangle \subseteq \beta\hat{\ }\langle\infty\rangle$. By our assumption on s_0, $\xi <_L \beta$, or $\xi \subset \beta$ but $\xi\hat{\ }\langle\infty\rangle \not\subseteq \beta$, is impossible. Again by our assumption on s_0 and since β is not in a block, $\xi \supseteq \beta\hat{\ }\langle\infty\rangle$ is impossible. And finally, by initialization at stage s_0, $\xi >_L \beta\hat{\ }\langle\infty\rangle$ is impossible (note that if such ξ performs $\mathcal{C}_x$-lifting then necessarily $\gamma(x) > s_0$ by $\mathcal{C}_x$-lifting at s_0). Thus β must enter a block $\mathcal{B}$ at some (least) stage $s^* \in [s_0, s]$, and by our assumption on s_0, it cannot be removed from $\mathcal{B}$, nor can $\mathcal{B}$ be discarded, by stage s.

Let $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ be the $\mathcal{M}$-strategies with $\beta_k\hat{\ }\langle\infty\rangle \subseteq \beta$ but $\beta_k \notin \mathcal{B}[s^*]$ (allowing $m = 0$). Then all these β_k have an expansionary stage $\in (s_0, s^*]$, so let s_k be the first β_k-expansionary stage $\in (s_0, s^*]$. We have $s_1 \leq s_2 \leq \cdots \leq s_m$. We set $s_{m+1} = s$ and claim

$$\begin{aligned} &\forall k \leq m \forall s' \in [s_k, s_{k+1}] \forall \xi \in T\, \forall j' \in J_{\mathcal{L}} \forall z \\ &\quad (\xi \text{ enumerates } z \leq s_k \text{ into } B_{j'} \text{ at } s' \rightarrow j' = j_0 \vee j' = j_{\mathcal{B}}). \end{aligned} \tag{13}$$

Suppose some $\xi \in T$ violates (13) at a (least) stage s'. Let $k \leq m$ be maximal such that $s' \in [s_k, s_{k+1}]$. Clearly, by the failure of (13), $\xi \subseteq \delta_{s'}$, and thus, by our assumption on s_0, $\xi\hat{\ }\langle\infty\rangle \subseteq \beta$ (and so ξ is an $\mathcal{M}$-strategy) or $\beta\hat{\ }\langle\infty\rangle <_L \xi$. We distinguish four subcases as follows:

Subcase 2a: $k = 0$ and $\beta\hat{\ }\langle\infty\rangle <_L \xi$: This is impossible by $\mathcal{C}_x$-lifting and initialization at $s_0 = s_k$.

Subcase 2b: $k = 0$ and $\xi\hat{\ }\langle\infty\rangle \subseteq \beta$: This is impossible since $s_0 = s_k$ is a ξ-expansionary stage, so if $z = \delta(y)[s'] \leq s_0$ then, by the minimality of s', $\Phi_\xi(A_{J_1^\xi}; y)$ and $\Phi_\xi(A_{J_2^\xi}; y)$ can be injured only via B_{j_0} or $B_{j_{\mathcal{B}}}$, and so ξ would correct via B_{j_0} or $B_{j_{\mathcal{B}}}$ at s'.

Subcase 2c: $k > 0$ and $\beta\hat{\ }\langle\infty\rangle <_L \xi$: By initialization at s_0, by $\xi \subseteq \delta_{s'}$, by the choice of k, and by Lemma 2(ii), we have that ξ is not eligible to act before s_k; so, by the failure of (13), ξ performs $\mathcal{C}_x$-lifting at s' by putting $z = \gamma(x)[s']$ into $B_{j'}$. This $\gamma(x)[s']$ must have been picked by some $\xi' >_L \xi$ before stage s_k; so, by $\mathcal{C}_x$-lifting at s_k and the choice of s_k, we must have $s' = s_k$. Let η be the longest common substring of β and ξ. Then, by $\xi \subseteq \delta_{s'}$, η must be an $\mathcal{M}$-strategy. By the choice of k, η is in a block $\mathcal{B}'$ at s_k; and it either performs $\mathcal{M}$-correction via $B_{j_{\mathcal{B}'}}$ at s_k (so ξ will use B_{j_0} to perform its $\mathcal{C}_x$-lifting at s'), or η (and any $\eta' \subset \eta$) does not enumerate any number (and thus ξ declares s_k a j_0-stage and performs $\mathcal{C}_x$-lifting via B_{j_0}). Thus in this subcase, ξ cannot violate (13).

Subcase 2d: $k > 0$ and $\xi\hat{\ }\langle\infty\rangle \subseteq \beta$: Since $\xi \subseteq \delta_{s'}$ and by the choice of k, we have $\xi \subseteq \beta_{k+1}$ (setting $\beta_{m+1} = \beta$). If there is a ξ-expansionary stage $\in [s_k, s']$ (e.g. if $\xi \subseteq \beta_k$) then for any $z = \delta_\xi(y)[s_k] < s_k$, we have $\varphi_\xi(y)[s_k] < s_k$, and so (13) implies that ξ cannot require $\mathcal{M}$-correction violating (13). So suppose there is no ξ-expansionary stage $\in [s_k, s']$ (i.e. assume $\xi = \beta_{k+1}$ for $k < m$ or $\xi \supset \beta_m$ for $k = m$). Then any $z = \delta_\xi(y)[s']$ must be $< s_0$ by the choice of s_{k+1}, thus $\varphi_\xi(y)[s'] < s_0$ and Subcase 2b establishes the desired contradiction.

We have thus established (13) in all four subcases. By the maximality of s_0 (as the greatest β-expansionary stage $\leq s'$), (13) (for $k = 0$) now establishes the lemma since for any $\delta(y)[s]$, we have $\varphi(y)[s] < s_0$ or $j_{\mathcal{B}} \in J_0$.

This concludes the proof of Lemma 6. ■

We can now verify the satisfaction of the $\mathcal{M}$-requirements:

Lemma 7 ($\mathcal{M}$-Satisfaction Lemma). *For each incomparable pair $(J_1, J_2) \in J^2_{\mathcal{L}}$ and each p.r. functional Φ, if $\Phi(A_{J_1}) = \Phi(A_{J_2})$ is total then there is a p.r. functional Δ such that $\Delta(A_{J_0}) = \Phi(A_{J_1})$ where $J_0 = J_1 \cap J_2$.*

Proof. Let $\beta \subset f$ be an $\mathcal{M}^{\Phi}_{J_1,J_2}$-strategy and $\Delta = \Delta_\beta$ the p.r. functional it builds. Suppose $\Phi(A_{J_1}) = \Phi(A_{J_2})$ is total. Then there are infinitely many β-expansionary stages and thus $\beta\hat{}\langle\infty\rangle \subset f$. Fix y. Once both $\Phi(A_{J_1}) \restriction (y+1)$ and $\Phi(A_{J_2}) \restriction (y+1)$ are defined with A_{J_1}- and A_{J_2}-correct computations, $\delta(y)$ can be increased at most once (for the sake of $\mathcal{M}$-correction); thus eventually $\Delta(A_{J_0}; y)$ is defined forever. By Lemma 6, if β requires $\mathcal{M}$-correction for y then β will destroy $\Delta(A_{J_0}; y)$ before the next β-expansionary stage; thus $\Delta(A_{J_0}; y) = \Phi(A_{J_1}; y)$ as desired. ■

This concludes the proof of our theorem.

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