HOMOMORPHISMS AND QUOTIENTS OF DEGREE STRUCTURES

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ABSTRACT. We investigate homomorphisms of degree structures with various relations, functions and constants. Our main emphasis is on pseudolattices, i.e., partially ordered sets with a join operation and relations simulating the meet operation. We show that there are no finite quotients of the pseudolattice of degrees or of the pseudolattice of degrees $\leq \mathbf{0}'$, but that many finite distributive lattices are pseudolattice quotients of the pseudolattice of computably enumerable degrees.

1. HOMOMORPHISMS

We will be proving some results on homomorphisms of degree structures. For some structures, there is already a body of knowledge; for others, little is known. We begin, in Section 1, with a discussion of homomorphisms in general, and homomorphisms of degree structures in particular, restricting our attention to homomorphisms which are onto maps, as these determine the quotients of the degree structures. In Section 2, we prove some easy results about quotients of certain degree structures. We then turn our attention to the degree structure which is the main emphasis of this paper, the pseudolattice $\mathcal{R}_p$ of computably enumerable (c.e.) degrees. In this setting, it is a corollary of a result of Calhoun [2] that every finite boolean algebra is isomorphic to a quotient of $\mathcal{R}_p$. We present a new proof of Calhoun's result in Sections 3 and 4, and use it as a model for further results. The proof of Calhoun's result relies on the fact that, in a boolean algebra, if a and b are meet-irreducible elements which are $\neq 1$, then they are incomparable, so the various requirements which need to be satisfied act almost independently of one another. This is no longer true when passing to distributive lattices, so a scheme must be found to get the various requirements to mesh. We have found a way to do this for those finite distributive lattices which satisfy an additional property, biorderability, a property which covers all (finite) linearly ordered sets. It is the aim of this paper to introduce new techniques which allow one to handle lattices in which there are comparable meet-irreducible elements, and our main theorem, stated and proved in Section 5, assumes that the lattice is biorderable, covering the most general situation which we can handle at this time.

Definition 1.1. Let $\mathcal{S}$ and $\mathcal{T}$ be structures with underlying universes S and T respectively, and fixed signature. The interpretation in a structure of symbols in

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the signature will be denoted by using the universe of the structure as a subscript for the symbol. A *homomorphism* $f : S \to T$ is a map f from S into T which satisfies the following conditions:

(i) For every constant symbol c, $f(c_S) = c_T$.

(ii) For every n-place relation symbol R and $a_1, \dots, a_n \in S$, if $\mathcal{S} \models R(a_1, \dots, a_n)$ then $\mathcal{T} \models R(f(a_1), \dots, f(a_n))$.

(iii) For every n-place function symbol g and $a_1, \dots, a_{n+1} \in S$, if $g_S(a_1, \dots, a_n) = a_{n+1}$ then $g_T((f(a_1), \dots, f(a_n)) = f(a_{n+1})$.

We will also refer to f as a homomorphism from $\mathcal{S}$ to $\mathcal{T}$.

We will consider signatures built from the following symbols.

- The constant symbol 0, denoting the least element of the structure.
- The constant symbol 1, denoting the greatest element of the structure.
- The binary relation symbol $\leq$, denoting the partial ordering of the structure.
- The binary function symbol $\vee$, denoting the join of two elements of the structure.
- The binary function symbol $\wedge$, denoting the meet of two elements of the structure.
- The unary function symbol $'$, denoting the complement of an element of the structure.
- The $n+1$-ary relation symbols M_n for $n \geq 2$, where $M_n(a_0, \dots, a_n)$ is to be interpreted as "every c which is $\leq a_i$ for every $i < n$ is also $\leq a_n$".

We now describe the structures which will be considered.

Definition 1.2. A *poset* is a structure in the language $\leq$ which satisfies the axioms for partially ordered sets. If we include the symbol 0 in our language, then the structure will be a *poset with least element*, if we include the symbol 1 in our language, then the structure will be a *poset with greatest element*, and if we include both 0 and 1 in our language then the structure will be a *poset with least and greatest element*. A *poset homomorphism* $f : A \to B$ is one which satisfies $a_1 \leq a_2 \to f(a_1) \leq f(a_2)$ for all $a_1, a_2 \in A$. If we have a poset with least (greatest, resp.) element then we require, in addition, a homomorphism f to map the least (greatest, resp.) element of A to the least (greatest, resp.) element of B.

Definition 1.3. An *upper semilattice (usl* for short) is a structure in the language $\leq$ and $\vee$ which is a poset having least upper bounds. While $\leq$ is definable from $\vee$, we include it for convenience. As before, the inclusion of 0 and/or 1 will add the designation *least* and/or *greatest element*. A *usl homomorphism* is a poset homomorphism $f : A \to B$ which also satisfies $f(a_1 \vee a_2) = f(a_1) \vee f(a_2)$ for all $a_1, a_2 \in A$. If the usl has least element 0 (greatest element 1, resp.) then we also require the homomorphism to satisfy $f(0) = 0$ ($f(1) = 1$, resp.). Our convention is that whenever a usl has a least (greatest, resp.) element, then 0 (1, resp.) is automatically included in the language, and the usl is treated as one with least (greatest, resp.) element.

Definition 1.4. A *lattice* is a structure in the language $\leq$, $\vee$ and $\wedge$ which is a poset having least upper bounds and greatest lower bounds. The inclusion of 0 and/or 1 will add the designation *least* and/or *greatest element*. A *lattice homomorphism* is a usl homomorphism $f : A \to B$ which also satisfies $f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$ for all $a_1, a_2 \in A$. If the lattice has least element 0 (greatest element 1, resp.) then we also require the homomorphism to satisfy $f(0) = 0$ ($f(1) = 1$, resp.). A lattice is *distributive* if it satisfies the standard distributive laws $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and its dual, and is *nondistributive* otherwise. Our convention is that whenever a lattice has a least (greatest, resp.) element, then 0 (1, resp.) is automatically included in the language, and the lattice is treated as one with least (greatest, resp.) element.

Definition 1.5. A *boolean algebra* is a structure in the language $\leq$, $\vee$, $\wedge$, $'$, 0 and 1 which is a distributive lattice with least and greatest elements $0 \neq 1$ in which each element has a complement. A *boolean algebra homomorphism* is a lattice homomorphism $f : A \to B$ which also satisfies $f(a') = f(a)'$ for all $a \in A$.

Definition 1.6. A *pseudolattice* is a structure in the language $\leq$, $\vee$ and $\{M_n : 2 \leq n < \omega\}$, i.e., a usl with the relations M_n interpreted as described above (the relations M_n are called *pseudomeet* relations). As before, the inclusion of 0 and/or 1 will add the designation *least* and/or *greatest element*. A *pseudolattice homomorphism* from a structure $\mathcal{A}$ with universe A to a structure $\mathcal{B}$ with universe B is a usl homomorphism $f : A \to B$ which also satisfies $\mathcal{A} \models M_n(a_0, \ldots, a_n) \to \mathcal{B} \models M_n(f(a_0), \ldots, f(a_n))$ for all $n \geq 2$ and all $a_0, \ldots, a_n \in A$.

We will primarily be interested in the possible homomorphic images (quotients) of degree structures. In particular, for given sets $\mathcal{U}$ of potential quotients, we will want to know the following about the structure $\mathcal{S}$: Is there a homomorphism of $\mathcal{S}$ onto some element of $\mathcal{U}$? Is there a homomorphism of $\mathcal{S}$ onto every element of $\mathcal{U}$? The first question is captured by the following properties.

Definition 1.7. Let $\mathcal{S}$ be a structure in a fixed signature, and let $\mathcal{U}$ be a class of structures in that same signature. We say that $\mathcal{S}$ is *$\mathcal{U}$-simple* if there is no homomorphism of $\mathcal{S}$ onto any element of $\mathcal{U}$, and call $\mathcal{S}$ *simple* if it is $\mathcal{U}$-simple for the class $\mathcal{U}$ of all non-trivial structures except for those isomorphic to $\mathcal{S}$. (For example, if the signature is that of pseudolattices with least and greatest elements, then the trivial structure is the one with a single element.) Of particular interest will be the set $\mathcal{F}$ of all non-trivial finite structures in the given signature.

Our study of pseudolattice homomorphisms will focus on *ideals* and *filters*.

Definition 1.8. Let $\mathcal{P}$ be a pseudolattice with universe P, and let $I \subseteq P$. I is an *ideal* of $\mathcal{P}$ (or of P) if the following conditions hold:

(i) If $a \in I$, $b \in P$ and $b \leq a$, then $b \in I$.

(ii) If $a, b \in I$ then $a \vee b \in I$.

An ideal I is *non-trivial* if $I \neq \emptyset$ and $I \neq P$. I is a *prime ideal* if it is non-trivial and also satisfies the following condition:

(iii) For any $a_0, \ldots, a_{n-1} \in P - I$ and $a_n \in I$, $\mathcal{P} \models \neg M_n(a_0, \ldots, a_n)$.

A *filter* is a set that is closed upwards. (Note that the definition of prime ideal follows standard algebraic guidelines as there is no meet in the language, but differs from the use in the literature of degrees. Algebraically, prime ideals are the ones which give rise to homomorphisms of the structure, and this motivated our choice of definition. Moreover, since there is no meet operation in our language, the algebraic definition of a filter in this setting would just be a subset that is closed upwards.)

2. Some easy results

We will mention some known results in this section, and also prove a few new simple results. Our main results will deal with $\mathcal{R}_p$, the pseudolattice with least and greatest elements whose universe is the c.e. degrees, and will be presented in later sections. We mention only the non-trivial results, noting that in the case of poset and usl homomorphisms any embedding of finitely many (principal) ideals trivially gives rise to a homomorphism onto the structure generated by those ideals under intersection; a degree is mapped into the smallest ideal in which it lies, and to 1 if it is not in any of the ideals.

We begin with a discussion of structures whose universe is the set $\mathbf{D}$ of all degrees. When endowed with poset structure, we have the following result of Slaman and Steel [7]: Under the hypothesis of the Axiom of Determinacy, for every poset homomorphism f such that $f(x) > x$ for all x, there is a degree c such that f, when restricted to the degrees $\geq c$, is an iterate of the jump operator. The only non-trivial result on usl homomorphisms of which we are aware appears in Rogers [6], where it is shown that the jump operator is not a usl homomorphism. We now show that $\mathcal{D}_p$, the pseudolattice with least element and universe $\mathbf{D}$, is $\mathcal{F}$-simple. The proof of the result relies on the next lemma. It is unknown whether $\mathcal{D}_p$ is simple.

Lemma 2.1. *Suppose that $\mathcal{M}$ and $\mathcal{L}$ are pseudolattices with least element, that $f : \mathcal{M} \to \mathcal{L}$ is an onto pseudolattice homomorphism where $\mathcal{M}$ and $\mathcal{L}$ have universes M and L respectively, that $a, b, c \in M$ with $a \vee b \geq c$, that $\mathcal{M} \models M_2(a, b, 0_M)$, and that $f(a) = f(b)$. Then $f(a) = f(b) = f(c) = 0_L$.*

Proof. Since $\mathcal{M} \models M_2(a, b, 0_M)$ and f is a pseudolattice homomorphism, it must also be the case that $\mathcal{L} \models M_2(f(a), f(b), f(0_M))$. Now $f(0_M) = 0_L$ and $f(a) = f(b)$, so $\mathcal{L} \models M_2(f(a), f(a), 0_L)$. As $f(a) \leq f(a)$, we conclude by the axiom defining M_2 that $f(a) \leq_L 0_L$ and hence that $f(a) = f(b) = 0_L$. But $a \vee b \geq c$ so

$$f(c) \leq f(a) \vee f(b) = 0_L \vee 0_L = 0_L.$$

□

Theorem 2.2. *$\mathcal{D}_p$ is $\mathcal{F}$-simple.*

Proof. Suppose that $\mathcal{L}$ is a finite pseudolattice of cardinality n with least element and universe L, and that f is a pseudolattice homomorphism from $\mathcal{D}_p$ onto $\mathcal{L}$. It is easily seen that every finite pseudolattice with least element is a lattice with greatest element. Fix $\mathbf{c} \in \mathbf{D}$ such that $f(\mathbf{c}) = 1_L$. By the methods of Jockusch and Posner [4], there is a set $\mathbf{M}$ of $n + 1$ minimal degrees, any pair of which have join $\geq \mathbf{c}$. By the Pigeonhole Principle, there are $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ such that $f(\mathbf{a}) = f(\mathbf{b})$. By Lemma 2.1, $f(\mathbf{c}) = 0_L$, so $0_L = 1_L$. □

We now turn to structures with universe $\mathbf{D}[\mathbf{0}, \mathbf{0}']$, the degrees below $\mathbf{0}'$. As is the case for $\mathbf{D}$, the jump operator and its iterates provide examples of non-trivial poset homomorphisms, but these are not usl homomorphisms. We now show that $\mathcal{D}_p[\mathbf{0}, \mathbf{0}']$, the pseudolattice with least and greatest elements whose universe is $\mathbf{D}[\mathbf{0}, \mathbf{0}']$, is $\mathcal{F}$-simple. It is unknown whether $\mathcal{D}_p[\mathbf{0}, \mathbf{0}']$ is simple.

Theorem 2.3. *$\mathcal{D}_p[\mathbf{0}, \mathbf{0}']$ is $\mathcal{F}$-simple.*

Proof. Suppose that $\mathcal{L}$ is a finite pseudolattice of cardinality n with least and greatest elements and universe L, and that f is a pseudolattice homomorphism from $\mathcal{D}_p[\mathbf{0},\mathbf{0}']$ onto $\mathcal{L}$. We have noted that every finite pseudolattice with least element is a lattice with greatest element; let 1_L be the greatest element in the universe L of $\mathcal{L}$, and note that, as f is an onto pseudolattice homomorphism, $f(\mathbf{0}') = 1_L$. By Lerman and Shore [5], there is a set $\mathbf{M}$ of $n+1$ minimal degrees, any pair of which have join $\mathbf{0}'$. By the Pigeonhole Principle, there are $\mathbf{a},\mathbf{b} \in \mathbf{M}$ such that $f(\mathbf{a}) = f(\mathbf{b})$. By Lemma 2.1, $f(\mathbf{0}') = 0_L$, so $0_L = 1_L$. □

We next consider structures with universe $\mathbf{D_2}$, the set of 2-c.e. degrees. As is the case for $\mathbf{D}$, the jump operator and its iterates provide examples of non-trivial poset homomorphisms, but these are not usl homomorphisms. We now show that $\mathcal{D}_{2,p}$, the pseudolattice with least and greatest elements whose universe is $\mathbf{D_2}$, is $\mathcal{F}$-simple. It is unknown whether $\mathcal{D}_{2,p}$ is simple.

Theorem 2.4. *$\mathcal{D}_{2,p}$ is $\mathcal{F}$-simple.*

Proof. Suppose that $\mathcal{L}$ is a finite pseudolattice of cardinality n with least and greatest elements and universe L, and that f is a pseudolattice homomorphism from $\mathcal{D}_{p,2}$ onto $\mathcal{L}$. We have noted every finite pseudolattice with least element is a lattice with greatest element; let 1_L be the greatest element in the universe L of $\mathcal{L}$, and note that, as f is an onto pseudolattice homomorphism, $f(\mathbf{0}') = 1_L$. By Downey [3], there is a set $\mathbf{M}$ of $n+1$ 2-c.e. degrees, any pair of which have join $\mathbf{0}'$ and meet $\mathbf{0}$. By the Pigeonhole Principle, there are $\mathbf{a},\mathbf{b} \in \mathbf{M}$ such that $f(\mathbf{a}) = f(\mathbf{b})$. By Lemma 2.1, $f(\mathbf{0}') = 0_L$, so $0_L = 1_L$. □

Now consider structures with universe $\mathbf{R}$, the set of c.e. degrees. As is the case for $\mathbf{D}$, the jump operator and its iterates provide examples of non-trivial poset homomorphisms, but these are not usl homomorphisms. Ambos-Spies, Jockusch, Shore and Soare [1] have shown that the cappable degrees (degrees $\mathbf{a}$ for which there is a $\mathbf{b} > \mathbf{0}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$) form a pseudolattice prime ideal $\mathcal{M}$ with complement $\mathcal{P}$ the set of prompt degrees. In fact, given two degrees $\mathbf{a},\mathbf{b} \in \mathcal{P}$, there is a $\mathbf{c} \in \mathcal{P}$ such that $\mathbf{c} \leq \mathbf{a},\mathbf{b}$. Thus we can define non-trivial homomorphisms of $\mathcal{R}_p$ onto $\mathcal{R}_p/\mathcal{M}$ and $\{0,1\}$. For the first structure, the homomorphism is a usl homomorphism, and for the second structure we have a pseudolattice homomorphism. In particular, $\mathcal{R}_p$ is not $\mathcal{F}$-simple, either as a usl or as a pseudolattice (both treated as structures with least and greatest elements).

For the remainder of this paper, we will focus on finite distributive quotients of $\mathcal{R}_p$ (treated as a pseudolattice with least and greatest elements). The scheme for defining homomorphisms is described below, and motivates the requirements to be imposed on the constructions of subsequent sections.

Let $\mathcal{M}$ and $\mathcal{L}$ be pseudolattices with least and greatest elements and universes M and L, respectively, and assume that L is a finite distributive lattice. We will attempt to define a pseudolattice homomorphism $f : M \to L$ as follows. For each $a \in L$, we will define a subset I_a of M. These sets will satisfy the following properties for all $d \in L$ and all meet-irreducibles b and c for which the sets in question are defined (b is meet-irreducible if $b = d \wedge e$ implies $d = b$ or $e = b$):

$$I_1 = M. \tag{2.1}$$

$$b \leq c \;\rightarrow\; I_b \subseteq I_c. \tag{2.2}$$

$$I_d - \bigcup\{I_a : a \ngeq d\} \neq \emptyset. \tag{2.3}$$

$$I_d = \bigcap\{I_b : b \geq d \;\&\; b \text{ is meet-irreducible}\}. \tag{2.4}$$

$$\begin{gathered}(b \neq 1 \;\&\; e_0, \ldots, e_{n-1} \notin I_b \;\&\; d_0, \ldots, d_m \in I_b \\ \& \; e_n = \vee\{d_i : i \leq m\}) \;\rightarrow\; \exists \widetilde{e}(\widetilde{e} \leq e_0 \;\&\ldots\&\; \widetilde{e} \leq e_{n-1} \;\&\; \widetilde{e} \nleq e_n).\end{gathered} \tag{2.5}$$

The next lemma shows that every pseudolattice homomorphism preserving least and greatest elements which maps onto a finite distributive lattice gives rise to sets having these properties. We use the following fact about finite distributive lattices L:

For every meet-irreducible $b \in L$ such that $b \neq 1$

there is a smallest $c \in L$ such that $c \nleq b$. (2.6)

To see (2.6) let $S = \{a : a \nleq b\}$ and $c = \wedge S$. Note that $S \neq \emptyset$ as $1 \in S$. We assume that $c \leq b$ and derive a contradiction. Under this assumption, $c \notin S$, so as S is finite, there are incomparable $d, e \in L$ such that $d, e \nleq b$ but $d \wedge e = c \leq b$. (To obtain d and e, let I be the set of minimal meet irreducibles $\geq c$. Since L is distributive, $c = \wedge I$ but $c \neq \wedge I'$ for any proper subset I' of I. Let d be any element of I and let $e = \wedge(I - \{d\})$.) As L is distributive, $b = b \vee (d \wedge e) = (b \vee d) \wedge (b \vee e)$. But $b \vee d, b \vee e \nleq b$, so b is not meet-irreducible, a contradiction.

Lemma 2.5. *Let $\mathcal{M}$ and $\mathcal{L}$ be pseudolattices with least and greatest elements and universes M and L, respectively, let $f : M \to L$ be an onto pseudolattice homomorphism preserving least and greatest elements, and assume that L is finite. Then there is a collection $\{I_a : a \in L\}$ of subsets of M satisfying (2.1)–(2.5).*

Proof. For each $a \in L$, define $I_a = \{d \in M : f(d) \leq a\}$. (2.1) and (2.2) are immediate, and (2.3) follows from the assumption that f is onto. By (2.1), the right hand side of (2.4) is not the intersection of the empty set, so (2.4) follows from the fact that every element d of a finite lattice is the meet of the meet-irreducibles $\geq d$. We assume that (2.5) fails and derive a contradiction. Fix notation as in (2.5) and assume that the failure is for the meet-irreducible element b. It suffices to consider the case where $n \geq 2$, as if $n = 1$ then (2.5) for the sequence $\langle e_0, e_1\rangle$ follows from (2.5) for the sequence $\langle e_0, e_0, e_1\rangle$. By (2.6), there is a smallest element $c \in L$ which is not $\leq b$. The failure of (2.5) implies that $\mathcal{M} \models M_n(e_0, \ldots, e_n)$. As f is a pseudolattice homomorphism, we must have $\mathcal{L} \models M_n(f(e_0), \ldots, f(e_n))$. We note that $f(e_0), \ldots, f(e_{n-1}) \geq c$ and $f(d_0), \ldots, f(d_m) \leq b$. As f preserves joins, $f(e_n) \leq b$ so $\mathcal{L} \models M_n(c, \ldots, c, b)$. We conclude that $c \leq b$, contrary to our assumption. □

Suppose that (2.1)–(2.5) hold. We define the map f by

$$f(d) = a \Leftrightarrow d \in I_a \;\&\; \forall b < a(d \notin I_b). \tag{2.7}$$

To see that f is well-defined, fix d, let $C = \{c : d \in I_c \;\&\; c \text{ is meet-irreducible}\}$, and let $a = \wedge C$. As L is distributive, and as, by (2.2), C is closed upwards, C must be the set of all meet-irreducibles $\geq a$ (since in a finite distributive lattice, every element is the meet of the set of minimal meet-irreducibles above it, but not the meet of any proper subset of this set). It now follows from (2.4) and the definition of f that $d \in I_c$ iff $c \geq a$, so f is well-defined. The next lemma shows that f is a pseudolattice homomorphism preserving least and greatest elements.

Lemma 2.6. *Let $\mathcal{M}$ and $\mathcal{L}$ be pseudolattices with least and greatest elements and universes M and L, respectively, and assume that L is a finite distributive lattice. Let f be defined as in (2.7), under the assumption that (2.1)–(2.5) hold. Then f is a pseudolattice homomorphism of $\mathcal{M}$ onto $\mathcal{L}$ preserving least and greatest elements.*

Proof. It follows from (2.3) that f is onto.

To see that f is order-preserving, fix $d, e \in M$ such that $d \geq e$ and let $f(d) = b$ and $f(e) = a$. Suppose that $a \not\leq b$ in order to obtain a contradiction. As every element $p \in L$ is the meet of all meet-irreducible elements $\geq p$, there must be a meet-irreducible $\widetilde{b} \in L$ such that $b \leq \widetilde{b}$ but $a \not\leq \widetilde{b}$. Note that by (2.4) and (2.7), for every $c \in M$, $f(c)$ is the smallest $p \in L$ such that $c \in I_p$. We conclude that $d \in I_{\widetilde{b}}$, and as $a \not\leq \widetilde{b}$, $e \notin I_{\widetilde{b}}$. By (2.5), there is an $\widetilde{e} \in M$ such that $\widetilde{e} \leq e$ but $\widetilde{e} \not\leq d$. Thus $e \not\leq d$, yielding the desired contradiction.

As f is onto and order-preserving, f must preserve least and greatest elements.

To see that f preserves joins, suppose that $d_1, d_2, e \in M$ and $d_1 \vee d_2 = e$. Let $a_1 = f(d_1)$, $a_2 = f(d_2)$ and $b = f(e)$. As f is order-preserving, $a_1, a_2 \leq b$ so $a_1 \vee a_2 \leq b$. Suppose that $a_1 \vee a_2 = c < b$ in order to obtain a contradiction. As in the preceding paragraph, there is a meet-irreducible element $\widetilde{c} \in L$ such that $c \leq \widetilde{c}$ but $b \not\leq \widetilde{c}$. By (2.5), there is a $d \in M$ such that $d \leq e$ but $d \not\leq d_1 \vee d_2$ contradicting the assumption that $d_1 \vee d_2 = e$.

Finally, we show that f preserves the pseudomeet relations. Suppose that $\mathcal{M} \models M_n(b_0, \ldots, b_n)$ but $\mathcal{L} \models \neg M_n(f(b_0), \ldots, f(b_n))$ in order to obtain a contradiction. Then there is a $c \in L$ such that $c \leq f(b_0), \ldots, f(b_{n-1})$ and $c \not\leq f(b_n)$. As above, there is a meet-irreducible element $a \in L$ such that $c \not\leq a$ and $f(b_n) \leq a$. Thus $b_n \in I_a$ and $b_0, \ldots, b_{n-1} \notin I_a$. By (2.5), $\mathcal{M} \models \neg M_n(b_0, \ldots, b_n)$, yielding the desired contradiction.

We conclude that f is a pseudolattice homomorphism from $\mathcal{M}$ onto $\mathcal{L}$ preserving least and greatest elements. □

3. Partition trees, requirements, and modules

Calhoun's [2] construction of a countable set of mutually incomparable prime ideals of $\mathcal{R}_p$ implies that any finite boolean algebra is a quotient of $\mathcal{R}_p$. We will present a different proof of this theorem, based on Calhoun's basic modules, which we find more amenable to generalization. The main change is to eliminate Calhoun's use of the Recursion Theorem and emphasize the appeal to König's Lemma, thereby reducing the level of the priority argument from $\mathbf{0^{(4)}}$ to $\mathbf{0''}$. We will then indicate how to modify the proof to handle certain finite quotients which are distributive lattices.

In this section, we will describe the basic module used to satisfy requirements. Instead of building ideals of $\mathbf{R}$ directly, we define sets $\widetilde{I}_b$ for each b in our boolean algebra B which are degree invariant and whose elements come from $\{W_i : i \in \omega\} \cup \{C_b : b \in B\}$ treated as a set of names; thus different names of the same c.e.

set are treated as different elements, and each such name W is placed in exactly one of $\widetilde{I}_b$ or its complement $\widetilde{F}_b$.

Requirements will be indexed by labeled trees and elements of the quotient lattice. We will not specify a guess as to which names W of c.e. sets lie in which sets $\widetilde{I}_b$ in advance; the labeled trees will anticipate all possibilities, and the outcomes for all requirements will, together with König's Lemma, determine the partition.

It will suffice to satisfy (2.1)–(2.5); we will do this by showing that the degree invariant sets $\widetilde{I}_b$ have similar properties. As a finite boolean algebra does not have comparable meet-irreducible elements unless one of the elements is 1, (2.2) is vacuous in this case. For each b in our boolean algebra, we will construct a set C_b which is placed in $\widetilde{I}_c$ iff $c \geq b$, thereby satisfying (2.3). Our constructions will automatically satisfy (2.4). Thus our major focus will be on requirements introduced in order to satisfy (2.5). Such requirements will focus on a single meet-irreducible element a and a single functional for each possible sequence from $\{W_0, \ldots, W_m\}$. The satisfaction of each such requirement is based on a partition of $\{W_0, \ldots, W_m\}$ into a degree-invariant set $\widetilde{I}_a$ and its complement, the degree-invariant set $\widetilde{F}_{a'}$; we will rely on König's Lemma to produce a fixed ideal for which all subrequirements are satisfied.

We begin with an example where the quotient lattice has just two elements. Thus we will only build two sets, I and F. We will need to decide, for each c.e. set W, whether W is to be placed into I or F.

Consider a requirement focusing solely on the c.e. set W. We assign higher priority to placing W into I. But we will not be able to determine, in advance, a guess as to the placement of W. We will construct c.e. sets D and D_Φ, and pair a primary requirement, $\Phi(W) \neq D$ which operates under the assumption that $W \in I$, with a secondary requirement $\Psi(\emptyset) \neq D_\Phi$, which operates under the assumption that $W \in F$ as forced by Φ and succeeds only if we can show that $D_\Phi \leq_T W$. If, for every Φ there is a Ψ such that the outcome of the corresponding module is that $\Phi(W) \neq D$, then it will be consistent to place $W \in I$. Otherwise, there will be a Φ such that for every Ψ we will satisfy $\Psi(\emptyset) \neq D_\Phi$ and be able to show that $D_\Phi \leq_T W$, so it will be consistent to place $W \in F$.

The basic module for a single set W is described in Example 1 below, ignoring the sets constructed to satisfy (2.3). We begin by trying to adopt a Friedberg-Mučnik strategy to satisfy the primary requirement $\Phi(W) \neq D$. However, we cannot restrain W, so the placement of a potential diagonalization witness into D will not aid with the satisfaction of the requirement should a small number enter W. On the other hand, were we to try to satisfy a secondary requirement of the form $\Psi(\emptyset) \neq E$ and $E \leq_T W$ using a Friedberg-Mučnik strategy, we would need W-permission of our diagonalization witness, and cannot force this to occur. The solution is to design a module which tries to satisfy these requirements sequentially. Thus before attempting to diagonalize for our primary requirement, we introduce a diagonalization witness for the secondary requirement which is larger than the Φ-use of the computation for the primary requirement. Only when both requirements are ready to act, do we diagonalize for the primary requirement. A change to W which injures this effort will provide the W-permission required for the secondary requirement to act. As the preservation of the success of the action by the secondary requirement does not require the restraint of any sets, one of our attempts will succeed.

The above description will generalize to an arbitrary situation. The basic module will be divided into two phases; the *design phase* during which we introduce the diagonalization witnesses (henceforth, these will be called *followers*) and wait for all appropriate computations to converge, and the *implementation phase*, during which we act to try to diagonalize, successively reacting to the permissions which occur.

Example 1: We present the basic module for a single Φ and Ψ.

Step 1: Choose a large *follower* p for the requirement $\Phi(W) \neq D$. (We will attempt to use p to establish $\Phi(W;p) \neq D(p)$.) Go to Step 2.

Step 2: Wait for a stage $s > 0$ at which $\Phi^s(W^s;p) \downarrow= 0$. Let u be the use of this computation. (While we are waiting, $D(p) = 0$; so if no such s is found during the course of the construction and infinitely many stages are considered, then the requirement will be satisfied.) We go to Step 3 when s is found. (We will be unable to act to diagonalize yet, as we must prepare for the contingency of a W-injury.)

Step 3: Choose a large *follower* $q \geq u$ for the requirement $\Psi(\emptyset) \neq D_\Phi$. (We will attempt to use q to establish $\Psi(\emptyset;q) \neq D_\Phi(q)$. By requiring that $q \geq u$, we are ensuring that a W-injury to the primary requirement will produce W-permission to place $q \in D_\Phi$.) Go to Step 4.

Step 4: Wait for a stage r at which either $\Psi^r(\emptyset;q) \downarrow= 0$ or $W^r \restriction u+1 \neq W^s \restriction u+1$. (While we are waiting, $D_\Phi(q) = 0$; so if no such r is found during the course of the construction and infinitely many stages are considered, then the requirement will be satisfied.) If we first find such a stage r, we go to Step 5. (The design phase will then end, and we will begin the implementation phase.) If we first find a stage r at which $W^r \restriction u+1 \neq W^s \restriction u+1$, then we return to Step 2, requiring that the new s is $> r$ in that step. (In this case, the W-injury is premature to be used to permit q, as we do not yet have a computation to diagonalize against. However, the follower p has not yet been used, so we can return to step 2 with the same follower p and try to find a new q for p. If this occurs infinitely often, then we will satisfy the primary requirement by showing that $\Phi(W;p) \uparrow$.)

Step 5: Place p into D^{r+1}. (We begin the implementation phase. Our action ensures that $\Phi^r(W^r;p) \downarrow= 0 \neq 1 = D^{r+1}(p)$. We do not control W, however, so cannot impose restraint on $W^r \restriction u+1$.) Pass to Step 6.

Step 6: Wait for a stage $t > r$ at which $W^t \restriction u+1 \neq W^s \restriction u+1$. (If t fails to exist, then Step 5 ensures the satisfaction of the requirement). If t is found, pass to Step 7.

Step 7: We place $q \in D_\Phi^{t+1}$. (Note that as $q \geq u$, this action is certified by W-permission.)

There are several possible outcomes for this module. If we wait in Step 2 forever or pass from Step 4 to Step 2 infinitely often, then either $\Phi(W;p) \uparrow\neq D(p)$ or $\Phi(W;p) \downarrow\neq 0 = D(p)$. If we wait in Step 4 forever, then either $\Psi(\emptyset;q) \uparrow\neq D_\Phi(q)$ or $\Psi(\emptyset;q) \downarrow\neq 0 = D_\Phi(q)$. If we wait in Step 6 forever, then $\Phi(W;p) \downarrow= 0 \neq 1 = D(p)$. Otherwise, we reach Step 7, so $\Psi(\emptyset;q) \downarrow= 0 \neq 1 = D_\Phi(q)$. Furthermore, we have permission from W to place q into D_Φ, so $D_\Phi \leq_T W$. Thus we will either satisfy the primary requirement, or we will satisfy the secondary requirement.

In general, W will be replaced by a finite collection $\mathcal{W}_n = \{W_0, \dots, W_n\}$ of c.e. sets. Any of the sets in $\mathcal{W}_n$ can cause the initial injury (the passage from Step 6 to Step 7), so we will have to anticipate every possibility. Injuries can occur to later requirements as well, if the oracle for the computation contains a set in $\mathcal{W}_n$. Thus we will introduce a tree to capture each possible sequence of injuries. We will not prioritize this tree, choosing instead to follow the sequence of injuries as they occur. However, we will need to linearize the nodes of this tree in order to carry out the design phase of the basic module. The trees will come equipped with a labeling which identifies the index of the computable partial functional associated with each node of the tree, and so identifies the subrequirement which we attempt to satisfy at that node. The trees are defined as follows.

Definition 3.1. The *n-tree* $\mathcal{T}_n = \langle T_n, \subseteq \rangle$ is the set of all sequences of elements from $\{0, 1, \dots, n\}$ without repetition, ordered by the end-extension relation $\subseteq$. A map h from T_n to the natural numbers is called a *labeling function* for T_n. The *labeled n-tree* $\mathcal{T}_n^h = \langle T_n, \subseteq, h \rangle$ consists of the n-tree $\mathcal{T}_n$ together with a labeling provided by h. If $\alpha \in T_n$ and h is a labeling of T_n, we set $h_\alpha = h \restriction \{\beta : \beta \subset \alpha\}$ and $h_\alpha^+ = h \restriction \{\beta : \beta \subseteq \alpha\}$. Note that T_n and hence T_n^h are finite. Let $\leq_\ell$ be the lexicographical ordering of T_n, and let $k(n)$ denote the cardinality of T_n.

Remark 3.2. If T_n^g and T_n^h are two labeled trees, $\alpha \in T_n$, and $g \supseteq h_\alpha$ then $g_\alpha = h_\alpha$; and if $g \supseteq h_\alpha^+$ then $g_\alpha^+ = h_\alpha^+$.

While the indexing of requirements will employ only labeled trees, there will be other outcomes for requirements which are not represented by the nodes of these trees. We thus pass to enhanced labeled trees, in order to represent all possible outcomes of requirements. If α is a node of a tree with immediate successor $\alpha^\frown\langle o \rangle$, we will abuse notation by calling both $\alpha^\frown\langle o \rangle$ and o *outcomes* for α. If $\alpha \neq \emptyset$, then α^- will denote the string $\beta \subset \alpha$ such that $|\beta| = |\alpha| - 1$. Nodes α in enhanced n-trees will have outcomes in $\{i : i \leq n\}$ which identify the set which causes the injury, in the implementation phase, to the requirement assigned to α; these outcomes are called *permission outcomes*. There will be additional outcomes, namely those in the set $\{w, c, r, z\}$. w, r and z are called *satisfaction outcomes*, and c is a *continuation outcome*. w will represent the outcome of the design phase in which we wait forever for a follower for the subrequirement assigned to α to be realized; c will represent the outcome of the design phase in which we pass to the next subrequirement; r will represent the outcome of the design phase in which we find a computation for a fixed follower of the subrequirement assigned to α infinitely often, but each time, an element enters the oracle to destroy the computation before we have had a chance to enter the implementation phase; and z will represent the outcome of the implementation phase in which we act to satisfy the subrequirement assigned to α, and its oracle is never injured thereafter.

Definition 3.3. The *enhanced n-tree* $\widetilde{\mathcal{T}}_n = \langle \widetilde{T}_n, \subseteq \rangle$ will be induced by the tree $\mathcal{T}_n$ and will contain $\emptyset$. Let $\langle \langle \alpha_i \rangle : i < k(n) \rangle$ list the elements of T_n in lexicographical order. For each $i < k(n)$ and $\alpha \in \widetilde{T}_n$ such that $|\alpha| = i$, α will be terminal if $|\alpha| > 0$ and either $\alpha = \alpha^{-\frown}\langle w \rangle$ or $\alpha = \alpha^{-\frown}\langle r \rangle$; otherwise, α will have outcomes w, c and r. Fix β such that $|\beta| = k(n)$ and $\beta(i) = c$ for all $i < k(n)$. If $\gamma \in T_n$, then $\beta^\frown\gamma$ and $\beta^\frown\gamma^\frown\langle z \rangle$ will be nodes of $\widetilde{T}_n$. This is a complete list of the nodes of $\widetilde{T}_n$. The finiteness of T_n implies that $\widetilde{T}_n$ is also finite. The lexicographical ordering $<_\ell$ on

$\widetilde{T}_n$ is induced by $\subseteq$ and the following orderings of outcomes;

$$0 <_\ell 1 <_\ell \cdots <_\ell n <_\ell z$$

and

$$r <_\ell c <_\ell w.$$

Each non-terminal $\alpha \in \widetilde{T}_n$ will be associated with a node $\gamma(\alpha)$ which gives rise to α. If $|\alpha| = i < k(n)$, then $\gamma(\alpha) = \alpha_i$; and if $\alpha = \beta^\frown\gamma$, then $\gamma(\alpha) = \gamma$. If h is a labeling of T_n, then the *enhanced labeled n-tree* $\widetilde{T}_n^h$ will be obtained from T_n^h by labeling all non-terminal nodes α of $\widetilde{T}_n^h$ with the label for $\gamma(\alpha)$ in T_n^h. Thus we set $h(\alpha) = h(\gamma(\alpha))$, $h_\alpha = h_{\gamma(\alpha)}$, and $h_\alpha^+ = h_{\gamma(\alpha)}^+$. (These definitions depend, not only on α, but on h and n; when they are used, we will have specified that α is to be chosen from a tree which specifies n and h, so the ambiguity which may appear in the definitions is resolved by the context.)

We now turn to a more general case. Let $\mathcal{B}$ be an arbitrary finite boolean algebra with universe B. For $a \in B$, we let a' denote the complement of a. We note that the meet-irreducible elements of B which are $\neq 1$ are just its coatoms, and that if a is a coatom, then a' is always the smallest element $\not\leq a$. For coatoms a, we let $F_{a'}$ denote the complement of the set I_a which we construct. In order to ensure that f is onto, we will construct a c.e. set C_a for each element $a \in B$, and require that $C_a \in I_b$ iff $a \leq b$ (note that b must be a coatom, as we are constructing I_b only in this case).

Remark 3.4. As we noted at the beginning of this section, it will suffice to concentrate on satisfying (2.5). The satisfaction of the following requirements for all coatoms $a \in B$ and all finite sets $F \subseteq F_{a'}$ and $I \subseteq I_a$ will suffice to ensure that f is a homomorphism:

$$R_{a,F,I}:\ \exists D(\forall\Phi(\Phi(\oplus\{W : W \in I\}) \neq D)\ \&\ (\forall V \in F(D \leq_T V))),$$

where the quantifier $\forall\Phi$ ranges over the computable partial functionals. In fact, we need not satisfy the above requirements for all finite I and F as described; rather, it suffices to ensure that for every I and F, there are $\widetilde{I} \subseteq I_a$ and $\widetilde{F} \subseteq F_{a'}$ such that $\widetilde{I} \supseteq I$, $\widetilde{F} \supseteq F$, and $R_{a,\widetilde{F},\widetilde{I}}$ is satisfied. (We will not be guessing at the choice of $\widetilde{I}$ and $\widetilde{F}$ for which we satisfy requirements; rather, we will try to satisfy the requirement for all possible choices, and rely on König's Lemma to sort out the final choice.)

As we will be using a tree of strategies at the $\mathbf{0}''$ level, we will have to utilize infinitely many lower level requirements. These will be indexed by coatoms $a \in B$, integers n, nodes of T_n and a partial labeling of T_n. Recall that we treat sets with different names from $\{W_i : i < \omega\} \cup \{C_b : b \in B\}$ as the different for the purpose of placement in I or F, even though they both may name the same subset of ω. Given $\alpha \in T_n$, we let $F_{\alpha,n} = \{W_{\alpha(i)} : i < |\alpha|\}$, $\widetilde{F}_{\alpha,n,a} = F_{\alpha,n} \cup \{C_d : d \geq a'\}$, $I_{\alpha,n} = \{W_i : i \leq n\} - F_{\alpha,n}$ and $\widetilde{I}_{\alpha,n,a} = I_{\alpha,n} \cup \{C_d : d \leq a\}$. Given $\alpha \in T_n$ and a labeling function h for T_n, the requirement assigned to α (which also depends on h) stipulates that we construct a c.e. set D_{α,n,a,h_α} for which the following holds:

$$R_{\alpha,n,a,h_\alpha^+}:\ \Phi_{h(\alpha)}(\oplus\widetilde{I}_{\alpha,n,a}) \neq D_{\alpha,n,a,h_\alpha}\ \&\ \forall V \in \widetilde{F}_{\alpha,n,a}(D_{\alpha,n,a,h_\alpha} \leq_T V).$$

(Note that by Remark 3.2, if $g \supseteq h_\alpha^+$ then $R_{\alpha,n,a,g_\alpha^+} = R_{\alpha,n,a,h_\alpha^+}$.) We will not try to satisfy this requirement, but rather will try to satisfy the requirement $R_{n,a,h}$

which stipulates that $R_{\alpha,n,a,h_\alpha^+}$ is satisfied for some $\alpha \in T_n^h$. Such a requirement is satisfied through the use of an $\langle n,a,h\rangle$-module, which we now define.

Definition 3.5. An $\langle n,a,h\rangle$-*module* is an algorithm based on an enhanced labeled n-tree $\widetilde{T}_n^h$ in which we follow the steps described below. Let $\{\alpha_i : i < k(n)\}$ list the elements of T_n^h, ordered lexicographically. We will refer to Case 1 as the *design phase,* and to Case 2 as the *implementation phase.* (The key idea in the ordering of steps in the design phase is that we ensure that any injury which occurs in the implementation phase at a given node α can be converted to a permission for the requirement assigned to the immediate successor of α which codes that injury.) The cases describe the action carried out for the non-terminal node $\alpha \in \widetilde{T}_n^h$ at stage s, and will refer to $\gamma(\alpha)$ as defined in Definition 3.3. We use the superscript s for sets and functionals to denote the value of those objects at the beginning of stage s, rather than at the end of stage s.

Case 1: $i < k(n)$. This case describes the *design phase* for α such that $\gamma(\alpha) = \alpha_i$.

We say that α *requires attention* at stage s if one of the following conditions holds at stage s:

$$\alpha \text{ does not have a follower.} \tag{3.1}$$

$$\alpha \text{ has an unrealized follower } p_i \text{ and } \Phi^s_{h(\alpha_i)}(\oplus\widetilde{I}^s_{\alpha_i,n,a};p_i)\downarrow= 0. \tag{3.2}$$

$$\alpha \text{ is not ripe, has a realized follower } p_i \text{ with oracle } \sigma_i \text{ and } \sigma_i \not\subseteq \oplus\widetilde{I}^s_{\alpha_i,n,a}. \tag{3.3}$$

If α does not require attention, then follow outcome w if the follower, p_i, of α is unrealized, and follow outcome c if p_i is realized.

Suppose that α requires attention. If (3.1) holds, we appoint a *follower* p_i and a *permission witness* q_i such that $p_i > q_i$ and both are greater than s and any number encountered in the construction before stage s and follow outcome w. p_i is *unrealized*. (We will attempt to use p_i to establish $\Phi_{h(\alpha_i)}(\oplus\widetilde{I}_{\alpha_i,n,a};p_i) \neq D_{\alpha_i,n,a,h_{\alpha_i}}(p_i)$, and will use q_i to force each C_d to permit p_i for $d \geq a'$.) If (3.2) holds, p_i becomes *realized* and we follow outcome c (passing to the next subrequirement). In this case, we let u_i be the *use* of the computation $\Phi^s_{h(\alpha_i)}(\oplus\widetilde{I}^s_{\alpha_i,n,a};p_i)$, define the *oracle* σ_i to be $\oplus\widetilde{I}^s_{\alpha_i,n,a} \restriction u_i+1$ and the *realization stage* to be $s_i = s$. If (3.3) holds, then p_i becomes *unrealized* and we follow outcome r (having injured the computation witnessing the realization of p_i). In this case, the use, oracle and realization stage are canceled. If we follow outcome c and $i = k(n)-1$, then we declare all nodes of the module to be *ripe* (signifying the completion of the design phase and start of the implementation phase).

If one of (3.1)–(3.3) holds at stage s, we say that α *receives attention* at stage s through the first of these equation numbers whose property holds.

Case 2: $i \geq k(n)$: Let $\gamma = \gamma(\alpha)$, and fix j such that $\gamma = \alpha_j$. This case describes the *implementation phase* for α.

We say that α *requires attention* at stage s if one of the following conditions holds at stage s:

$$p_j \text{ has not yet been released.} \tag{3.4}$$

$$p_j \text{ has been released and is uninjured, and } \sigma_j \not\subseteq \oplus\widetilde{I}^s_{\alpha_j,n,a}. \tag{3.5}$$

If α does not require attention, then follow outcome z for α if p_j is uninjured, and follow outcome k for α if p_j is k-injured.

Suppose that α requires attention at stage s. If (3.4) holds, then place p_j into $D^{s+1}_{\alpha_j,n,a,h_{\alpha_j}}$ and q_j into C^{s+1}_d for all $d \geq a'$ and follow outcome z for α. We say that p_j is *released*, and also that p_j is *uninjured*.

Suppose that (3.5) holds. p_j becomes *injured* at stage s. Fix the least $k \leq n$ such that $W_k \in \widetilde{I}^s_{\alpha_j,n,a}$ and $W^s_k \restriction u_j+1 \neq W^{s_j}_k \restriction u_j+1$; if no such k exists, then the construction is *terminated*. (It will follow from Lemma 4.4(iii) that the true path through our full tree of strategies is infinite, and so that such a k must exist.) If k exists, then we say that p_j becomes k*-injured* at stage s, and we follow outcome k for α.

If either (3.4) or (3.5) holds at stage s, we say that α *receives attention* at stage s through the first of these equation numbers whose property holds.

Let $\alpha = \alpha^{-\frown}\langle o\rangle$ be a terminal node of this module. If we follow α at all sufficiently large stages at which α^- is followed and $o = w$ or follow α infinitely often and $o = r$ for some α of shortest length, then, fixing i such that $\gamma(\alpha^-) = \alpha_i$, there is a stage at which p_i is appointed and never canceled thereafter, and $\Phi_{h(\alpha_i)}(\oplus\widetilde{I}_{\alpha_i,n,a}; p_i) \uparrow\neq D_{\alpha_i,n,a,h_{\alpha_i}}(p_i)$. If we follow α at all sufficiently large stages at which α^- is followed and $o = z$, and fix i such that $\gamma(\alpha^-) = \alpha_i$, then it will be the case that $\Phi_{h(\alpha_i)}(\oplus\widetilde{I}_{\alpha_i,n,a}; p_i) \downarrow= 0 \neq 1 = D_{\alpha_i,n,a,h_{\alpha_i}}(p_i)$. Note that these are the only possibilities, as a permission outcome or outcome c cannot be the last outcome of a module.

Conflicts between modules can occur only when one module places, into some C_a, a number which is smaller than the C_a-use of a computation that a second module wishes to preserve. The standard Π_2-priority tree construction will handle such conflicts through initialization as long as a given node of the tree acts in this way only finitely often. But this must be the case, as no number is placed into a set C_a when the infinitary outcome r is followed.

Note that these modules prescribe an order for straightforwardly generalizing the simpler module of Example 3.1 while incorporating a strategy for placing numbers into the sets $\{C_a : a \in B\}$. In the design phase, the order of action through the tree ensures that if $\alpha \subset \delta \in T^h_n$, then the design phase for α is completed before we start the design phase for δ. Hence any k-injury during the implementation phase to α can be used as a k-permission for δ. The set of nodes $\gamma(\alpha)$ for those α which receive attention during the implementation phase forms a non-decreasing sequence, under inclusion, as a function of the stage s, so the union of this sequence (viewed as a sequence of sets) produces a node of T^h_n.

4. The construction and proof; boolean algebras

We fix an effective *priority ordering* $\langle R_j : j < \omega\rangle$ of all requirements. Our tree of strategies T is a level 2 or $\mathbf{0}''$ tree. These trees will be built by gluing basic modules to $\emptyset$ and to terminal nodes of earlier basic modules, always choosing the highest priority module which has not yet been assigned to a predecessor of the node. Given $\sigma \in T$, there will be a unique non-terminal node ρ of a module such that σ corresponds to ρ; we define $\gamma(\sigma) = \gamma(\rho)$. T comes equipped with a priority ordering $<_\ell$ which is induced lexicographically under $\subseteq$ from the orderings $<_\ell$ defined for the outcomes of the basic modules. A *path* through the tree will be an infinite sequence coded by the outcomes of the nodes of the modules.

The Construction

Stage 0: All nodes are initialized. We set $\lambda_0 = \emptyset$.

Stage $\mathbf{s > 0}$: Case 1: There is an uninitialized σ which requires attention through (3.5) at stage s. Choose such a node σ of highest priority and carry out the instructions specified in the module of Definition 3.5. We say that σ *receives attention* at stage s. Let o be the outcome of σ at stage s and immediately uninitialize $\sigma^\frown\langle o\rangle$. We now implement action for $\rho = \sigma^\frown\langle o\rangle$ as specified by the construction for nodes satisfying (3.4), say that ρ *receives attention* at stage s, and set $\lambda_s = \rho^\frown\langle z\rangle$ which we uninitialize.

Case 2: Otherwise. We compute λ_s by determining a finite strictly increasing sequence of nodes $\langle \eta_i^s\rangle$ through the tree. Set $\eta_0^s = \emptyset$. Suppose that η_i^s has been defined. We carry out the instructions specified in the basic module of Definition 3.5 for η_i^s, yielding a path $\eta_{i+1}^s = \eta_i^{s\frown}\langle o\rangle$ of length $i+1$. We continue in this manner by induction, until we reach an initialized node τ. If τ is a node of the design phase of its module, then we set $\lambda_s = \tau$ and uninitialize τ. Otherwise, τ is a non-terminal node of the implementation phase of its module which is currently initialized; we immediately uninitialize τ. We now implement action at τ as specified by the construction for nodes satisfying (3.4), say that τ *receives attention* at stage s, set $\lambda_s = \tau^\frown\langle z\rangle$ and uninitialize λ_s. During this inductive process, each $\eta_i \subset \lambda_s$ which requires attention at stage s is said to *receive attention* at stage s.

In both cases, we *initialize* all nodes of lower priority than λ_s (canceling all numbers, strings and designations associated with these nodes), and go to the next stage. We call λ_s the *current path* at stage s. If the construction is terminated at stage s, we set all parameters for the construction at stages $t > s$ equal to the values of the corresponding parameters at stage s.

We make the following observation about the construction.

Remark 4.1. Suppose that $\eta \subseteq \lambda_s$ is a non-terminal node of the implementation phase of the copy M of a module on T and η receives attention at stage s. If η receives attention via (3.4) at stage s, then either η is initialized at the beginning of stage s or we set $\lambda_s = \eta^\frown\langle z\rangle$; and if η receives attention via (3.5) at stage s and $\lambda_s \supseteq \eta^\frown\langle i\rangle$, then $\eta^\frown\langle i\rangle$ is initialized at the beginning of stage s. Hence either $\lambda_s = \eta^\frown\langle z\rangle$, or there is an outcome $o \neq z$ such that $\lambda_s = \eta^\frown\langle o\rangle^\frown\langle z\rangle$ and $\eta^\frown\langle o\rangle$ is a non-terminal node of the implementation phase of M.

We define the *true path* Λ for the construction by induction. We begin by setting $\emptyset \subseteq \Lambda$. Suppose that we have specified that $\alpha \subseteq \Lambda$. It will then be the case that $|\{s : \alpha \subseteq \lambda_s\}| = \infty$. Fix the highest priority outcome $\alpha^\frown\langle o\rangle$, if any, for α such that $\{s : \alpha^\frown\langle o\rangle \subseteq \lambda_s\}$ is infinite, and specify that $\alpha^\frown\langle o\rangle \subseteq \Lambda$. It will follow from Lemma 4.4(iii) that $|\Lambda| = \infty$. Before proving that lemma, we will prove several lemmas which will analyze the local behavior of the construction within a fixed module when initialization does not occur for that portion of the module. The first of these lemmas provides an analysis of the implementation phase; it tells us that new action of the construction for non-terminal nodes in the implementation phase produces successively longer non-terminal nodes of that module.

Lemma 4.2. *Suppose that β is the node of a copy M of a module on T at which the implementation phase of M begins, that $\lambda_v \supseteq \beta$ but β is initialized at the end of stage $v-1$, and that $w > v$ is such that β is uninitialized at all stages t such that $v \leq t \leq w$. Then:*

(i) If $v < s \leq w$ and η is a non-terminal node of M which receives attention at stage s, then $\eta \supseteq \beta$.

(ii) If $v \leq s \leq w$, $\eta \supseteq \beta$ is a non-terminal node of M and $\lambda_s \supseteq \eta$, then η is uninitialized at all stages t such that $s \leq t \leq w$. Furthermore, if $\lambda_w \supseteq \beta$, then $\lambda_w \supset \eta$.

(iii) If $v \leq t \leq w$ and $\eta \supseteq \beta$ is a non-terminal node of M which is uninitialized at the end of stage t, then there is exactly one outcome o such that $\eta^\frown\langle o\rangle$ is uninitialized at the end of stage t. Furthermore, if $o \neq z$, then η does not receive attention at stage t, and its follower p_j is o-injured, and released at the end of stage t; and if $o = z$, then p_j is released and uninjured at the end of stage t.

Proof. (i): Suppose that $v < s \leq w$ and η is a non-terminal node of M which receives attention at stage s. We assume that $\eta \not\supseteq \beta$ and derive a contradiction. We note that the only non-terminal nodes of M which are not $\supseteq \beta$ are of the form $\xi^\frown\langle c\rangle$ with $\xi \subset \beta$. Without loss of generality, we assume that s is the smallest counterexample to (i) and that η is the shortest counterexample to (i) at s; thus $\eta^\frown\langle c\rangle \subseteq \beta$. By the construction and as $\eta^\frown\langle c\rangle \subseteq \beta \subseteq \lambda_v$ and β is uninitialized at all stages t such that $v \leq t < s$, $\eta^\frown\langle c\rangle$ must be uninitialized at stage $s-1$. As η receives attention at stage s, it can only be the case that $\eta^\frown\langle r\rangle \subseteq \lambda_s$. But then β is initialized at stage s, contrary to hypothesis.

We now prove (ii) and (iii) by induction on $w \geq v$ and then by induction on $|\eta|$ for $\eta \supseteq \beta$.

(ii): Suppose that $v \leq s \leq w$, $\eta \supseteq \beta$ is a non-terminal node of M and $\lambda_s \supseteq \eta$. (ii) follows easily from hypothesis, the construction and Remark 4.1, if either $v = w$ or $\eta = \beta$. Suppose that $w > v$. Let $\xi = \eta^-$. By hypothesis, every $\rho \subseteq \beta$ is uninitialized at stage w; thus by (ii) inductively, both ξ and η are uninitialized at stage $w-1$, and ξ is uninitialized at stage w. It now follows from the construction that η can be initialized at stage w only if some $\rho \supseteq \xi$ such that $\rho \not\supseteq \eta$ receives attention at stage w. If $\lambda_w \not\supseteq \xi$, then no such ρ cannot receive attention at stage w, so the conclusion of (ii) holds in this case.

It remains to consider the case wherein $\lambda_w \supseteq \xi \supseteq \beta$. By (iii) inductively, ξ has a unique uninitialized outcome o at the end of stage $w-1$. As η is uninitialized at the end of stage $w-1$ and $\eta^- = \xi$, we must have $\xi^\frown\langle o\rangle = \eta$; and as η is a non-terminal node of M, $o \neq z$. By (iii) inductively, ξ will not receive attention at stage w, so as $\lambda_w \supseteq \xi$, it follows from the construction that $\lambda_w \supseteq \eta$. Now by Remark 4.1, λ_w cannot be a non-terminal node of the implementation phase of a module, so $\lambda_w \supset \eta$. Thus η is uninitialized at stage w.

(iii): Let s be the smallest stage $\geq v$ at which η is uninitialized. Then as β, and hence $\eta \supseteq \beta$, is initialized at the end of stage $v-1$ and thus the follower of η cannot be released at the end of stage $s-1$, it follows from the construction that $\eta^\frown\langle z\rangle$ is the unique uninitialized immediate extension of η at stage s and the follower p_j of η is released and uninjured at the end of stage s. By induction, we assume that the lemma holds for η at all stages t such that $v \leq t < w$, and so there must be a unique outcome $\widehat{o}$ such that $\eta^\frown\langle\widehat{o}\rangle$ is uninitialized at the end of stage $w-1$; furthermore, for all ρ such that $\beta \subseteq \rho \subset \eta$, ρ is uninitialized at stage w and does not receive attention at stage w. Thus as $\lambda_w \supset \eta$, (iii) will hold unless η receives attention at stage w; assume that this is the case.

If $\eta^\frown\langle z\rangle$ is the unique uninitialized extension of η at stage $w-1$ and η receives attention at stage w, then the follower of η will be released before the beginning of stage w, and at stage w, we will initialize $\eta^\frown\langle z\rangle$, uninitialize a unique outcome o, and follow $\eta^\frown\langle o\rangle$; thus the lemma will hold for w in this case. If $\widehat{o} \neq z$, then the follower of η will have been released and η will have been o-injured at the end of stage $w-1$, so as η^- does not receive attention at stage w, η cannot receive attention at stage w. Hence (iii) holds. □

We now prove a lemma which implies that injuries to a computation cannot be caused by changes to the sets C_d.

Lemma 4.3. *Let σ be a node of a copy M of the design phase of a module for requirement $R_{\gamma(\sigma),n,a,h^+_{\gamma(\sigma)}}$, and fix i such that $\gamma(\sigma) = \alpha_i$. Suppose that we have stages $s < v$ such that $\sigma^\frown\langle c\rangle$ is initialized at the end of stage $s-1$ but $\sigma^\frown\langle c\rangle \subseteq \lambda_s$ and we do not act to cancel σ^s_i at any stage t such that $s \leq t \leq v$. Then for all $d \leq a$, $C^v_d \upharpoonright u^s_i + 1 = C^s_d \upharpoonright u^s_i + 1$.*

Proof. Suppose that the lemma fails, in order to obtain a contradiction. Then there is a $d \leq a$ and a number $q \in C^v_d - C^s_d$ such that $q \leq u^s_i$. Let q be placed in C^{t+1}_d as part of the action for node ρ. This can only happen if $s \leq t < v$ and ρ receives attention at stage t through (3.4). We note that ρ must be a node of the implementation phase of a module in order to require attention through (3.4), while σ is a node of the design phase of its module, so $\rho \neq \sigma$; and if $\rho \subset \sigma$, then ρ and σ are in different modules. As $s \leq t < v$, we now see that ρ cannot have higher priority than σ else we would cancel σ^s_i during stage t, contrary to hypothesis.

We cannot have $\rho \supseteq \sigma^\frown\langle w\rangle$ and ρ cannot have lower priority than $\sigma^\frown\langle w\rangle$, as all such nodes are initialized at stage s and any numbers they later cause to be placed into sets are $> u^s_i$. We cannot have $\rho \supseteq \sigma^\frown\langle r\rangle$, as then we would act to cancel σ^s_i at stage t. It remains to consider the case wherein $\rho \supseteq \sigma^\frown\langle c\rangle$. If ρ is a node of the module M, then ρ never acts to place numbers into C_b for any $b \leq a$; and if ρ is a node of a module different from M, then the node appointing q is initialized at stage s so the numbers it appoints (q in particular) are $> u^s_i$. Thus we have obtained a contradiction in all cases. □

(Lemma 4.3 is the crucial lemma for generalizing the homomorphism theorem of this section. In these generalizations, it will be important to merge the steps in the design and implementation phases for several modules into a single hybrid module in a way that allows us to obtain the conclusion of the lemma. All portions of the above proof will carry over, except for the case in which $\rho \supseteq \sigma^\frown\langle c\rangle$ and ρ is a node of the module M. We will need to place conditions on lattices which allow us to handle the latter case.)

Lemma 4.4. *Suppose that $\sigma \subseteq \Lambda$. Then:*

(i) There is a stage s_0 such that for all $t > s_0$, λ_t does not have higher priority than σ.

(ii) There is a stage $s_1 \geq s_0$ such that for all $t > s_1$, σ is uninitialized throughout stage t.

(iii) $\sigma \subset \Lambda$.

(iv) For every path Γ through T and any requirement $R_{n,a,h}$, there is a node $\tau \subset \Gamma$ such that a copy of the module for $R_{n,a,h}$ has initial node τ.

Proof. (i): Immediate from the definition of Λ and our assumption that $\sigma \subseteq \Lambda$.

(ii): Fix $s_1 \geq s_0$ such that $\sigma \subseteq \lambda_{s_1}$, where s_0 is chosen to satisfy (i). Then σ will be uninitialized at the end of stage s_1. The construction can act to initialize σ at stage $t > s_1$ only if λ_t has higher priority than σ; and as $s_1 \geq s_0$, this will never occur.

(iii): Fix s_1 for σ as in (ii), and fix notation as in the construction. It follows from Lemma 4.3 that (no C component of $\sigma_j^{s_j}$ could have changed, and hence) if $s \geq s_1$ and $\sigma_j^{s_j} \not\subseteq \oplus\widetilde{I}^s_{\alpha_j,n,a}$, then there must be a k such that $W_k \in I_{\alpha_j,n,a}$ and $W_k^s \restriction u_j^{s_j} + 1 \neq W_k^{s_j} \restriction u_j^{s_j} + 1$. Thus the construction is never terminated. Now by (ii) and as $\sigma \subseteq \Lambda$, there will be infinitely many $t > s_1$ at which $\lambda_t \supseteq \sigma$ and for each such t, σ will be uninitialized at the beginning of stage t. As the construction is never terminated and as, by Lemma 4.2(iii), λ_t is never set equal to a node which is uninitialized at the beginning of stage t, $|\{t : \lambda_t \supset \sigma\}| = \infty$. As σ has only finitely many possible outcomes, (iii) now follows from the Pigeonhole Principle.

(iv): As each module is finite, this condition follows immediately from the procedure for assigning modules to T. □

The next lemma shows that we must follow Case 2 of the construction infinitely often along Λ. The idea of the proof is as follows. By Lemma 4.4(iii) we will follow some case along Λ infinitely often, and by the construction, Case 1 can be followed only when some node requires attention via (3.5). Only nodes of the implementation phase of a module can require attention via (3.5), and in that case the follower of such a node must already be released. When this occurs for τ, all lower priority node are initialized with the exception of $\tau^\frown\langle i\rangle$ and $\tau^\frown\langle i\rangle^\frown\langle z\rangle$, where i is the number such that τ becomes i-injured. Thus the only node which may require attention at the next stage but was not eligible to require attention when τ received attention is $\tau^\frown\langle i\rangle$. As each path through a module is finite, the conclusion will follow.

Lemma 4.5. *Fix $\sigma \subset \Lambda$. Then*

$$|\{t : \sigma \subset \lambda_t \ \&\ \textit{Case 2 of the construction is followed at stage } t\}| = \infty.$$

Proof. Fix a stage s. By Lemma 4.4(ii),(iii), we may assume without loss of generality that $\lambda_s \supset \sigma$ and σ is uninitialized at all $t \geq s$. Hence a node $\supseteq \sigma$ receives attention at stage s. We note that if Case 1 of the construction is followed for τ at stage t, then τ requires attention at stage t via (3.5) and so the follower of τ has been released and is uninjured at the beginning of stage t and τ is a node of the implementation phase of its module, so $\tau^\frown\langle z\rangle$ is uninitialized at the end of stage $t-1$. For each $t \geq s$, let V^t be the set of all τ for which $\tau^\frown\langle z\rangle$ is uninitialized at the beginning of stage t, and let V^t_τ be the set of all nodes in V^t having higher priority than τ.

Suppose that $\tau \in V^t$ receives attention at stage $t \geq s$ through Case 1 of the construction. Then (3.5) holds for τ at stage t, and there is an i for which τ becomes i-injured and $\lambda_t \supset \tau^\frown\langle i\rangle$. Furthermore, by Lemma 4.2(iii) and as $\tau^\frown\langle z\rangle$ is uninitialized at the beginning of stage t, $\tau^\frown\langle i\rangle$ will be initialized at the beginning of stage t. By the construction, we will immediately uninitialize $\tau^\frown\langle i\rangle$, so (3.4) will apply to this node and we will set $\lambda_t = \tau^\frown\langle i\rangle^\frown\langle z\rangle$. By the initialization process, the only nodes of lower priority that τ that will be uninitialized at the end of stage t are $\tau^\frown\langle i\rangle$ and $\tau^\frown\langle i\rangle^\frown\langle z\rangle$. It thus follows that $V^{t+1} = V^t_\tau \cup \{\tau^\frown\langle i\rangle\}$.

We now see that whenever Case 1 of the construction is followed at stage t, the set of modules on T represented by elements of V^{t+1} is a subset of the set of modules on T represented by elements of V^t, and if $\rho \in V^{t+1} - V^t$, then $\rho^- \in V^t - V^{t+1}$, so we replace a node of module M which lies in V^t with one of its proper extensions, also lying in M. As any path through a module is finite, it follows from Lemma 4.4(iii) that it is impossible for the construction to follow Case 1 at all $t \geq s$ such that $\lambda_t \supset \sigma$, and the lemma follows. □

The next lemma will allow us to conclude that if a computation is declared at a node σ of the design phase of a module M and $\sigma^\frown\langle c\rangle$ is not initialized before we reach the node β beginning the implementation phase for M, then that computation remains valid when we reach β.

Lemma 4.6. *Suppose that σ is a node of the design phase of a copy M, on T, of a module for requirement $R_{\gamma(\sigma),n,a,h^+_{\gamma(\sigma)}}$, that $\lambda_v \supseteq \beta \supseteq \sigma^\frown\langle c\rangle$ where β is the node of M at which the implementation phase begins, and that β is initialized at the end of stage $v-1$. Fix i such that $\gamma(\sigma) = \alpha_i$. Then there is a smallest stage $s \leq v$ such that both $\lambda_s \supseteq \sigma^\frown\langle c\rangle$ and for all t such that $s \leq t \leq v$, λ_t does not have higher priority than $\sigma^\frown\langle c\rangle$. Furthermore, $\sigma^s_i \subseteq \oplus\widetilde{I}^v_{\alpha_i,n,a}$.*

Proof. We note that if Case 1 of the construction is followed to revise the extension of η at stage t, then η corresponds to a non-terminal node of the implementation phase of a module. As β^- is part of the design phase of M, Case 2 of the construction must be followed at stage v and $\sigma^\frown\langle c\rangle \subseteq \beta$ must be uninitialized at the end of stage v. Hence there is a smallest stage $s_0 \leq v$ such that $\sigma^\frown\langle c\rangle$ is uninitialized at the end of stage t for every t for which $s_0 \leq t \leq v$, and $s = s_0$ is the stage required by the lemma; furthermore as σ is a node of the design phase of its module and receives attention at stage s_0, Case 2 of the construction is followed at stage s. By choice of s, (3.3) fails to hold at any stage t such that $s \leq t \leq v$, $\sigma \subseteq \lambda_t$ and Case 2 is followed at stage t, else we would set $\lambda_t = \sigma^\frown\langle r\rangle$ which would have higher priority than $\sigma^\frown\langle c\rangle$ and initialize $\sigma^\frown\langle c\rangle$, contrary to hypothesis. Thus as Case 2 is followed at stage v, we must have $\sigma^s_i \subseteq \oplus\widetilde{I}^v_{\alpha_i,n,a}$. □

We now show that if certain permissions occur too soon for a non-terminal node in the implementation phase of a module, then the current path will not extend that node without first canceling the oracle set giving rise to the permission.

Lemma 4.7. *Let ξ be a non-terminal node of the implementation phase of a copy M of a module for requirement $R_{\gamma(\xi),n,a,h^+_{\gamma(\xi)}}$ on T. Fix σ in the design phase of M such that $\gamma(\xi) = \gamma(\sigma)$, fix i such that $\gamma(\sigma) = \alpha_i$, and fix j such that W_j is a component of $\widetilde{I}_{\alpha_i,n,a}$. Let β be the node of M at which the implementation phase begins. Suppose that $s < \widehat{s}$ are stages such that $\sigma^\frown\langle c\rangle$ is uninitialized at every stage t such that $s \leq t \leq \widehat{s}$. Suppose furthermore that $s \leq \widetilde{s} \leq \widehat{s}$ and j are given such that $W^{\widetilde{s}}_j \restriction u^s_i + 1 = W_j \restriction u^s_i + 1$ and that β is initialized at stage $\widetilde{s}$. Then $\xi^\frown\langle j\rangle \not\subseteq \lambda_t$ for any t such that $\widetilde{s} \leq t \leq \widehat{s}$.*

Proof. Suppose that such a t exists in order to obtain a contradiction, and fix the least such t. Then there must be a least $v \leq t$ such that $\lambda_v \supseteq \beta$ and β is uninitialized at all stages $\widetilde{t}$ such that $v \leq \widetilde{t} \leq t$. By Lemma 4.6, $\sigma^s_i = \sigma^s_i \restriction u^s_i + 1 \subseteq \oplus\widetilde{I}^v_{\alpha_i,n,a}$.

As $\sigma^\frown\langle c\rangle$ is not initialized at any stage between s and $\widehat{s}$, we will have $u_i^t = u_i^s$ and since $\widetilde{s} < t$, (3.5) cannot hold for ξ and j at stage t. $\square$

The next lemma shows that if we follow outcome $\sigma^\frown\langle w\rangle$ or $\sigma^\frown\langle r\rangle$, then the requirement assigned to σ is satisfied.

Lemma 4.8. *Suppose that $\sigma \subset \Lambda$ is a node which is part of the design phase of a module for requirement $R_{\gamma(\sigma),n,a,h^+_{\gamma(\sigma)}}$. Fix i such that $\gamma(\sigma) = \alpha_i$. Then:*

(i) There is a stage s and a follower p_i which has been assigned to σ by stage s and not canceled at any stage $t \geq s$.

(ii) If $\sigma^\frown\langle w\rangle \subset \Lambda$, then $\Phi_{h(\alpha_i)}(\oplus\widetilde{I}_{\alpha_i,n,a}; p_i) \neq D_{\alpha_i,n,a,h_{\alpha_i}}(p_i)$.

(iii) If $\sigma^\frown\langle r\rangle \subset \Lambda$, then $\Phi_{h(\alpha_i)}(\oplus\widetilde{I}_{\alpha_i,n,a}; p_i)\uparrow$.

Proof. (i): By Lemma 4.4(i),(iii), there is a stage s such that for all $t \geq s$, σ has higher priority than λ_t. By Lemma 4.5, we can assume that $\lambda_s \supset \sigma$ and that Case 2 of the construction is followed at stage s. As σ would require attention, and hence would receive attention through (3.1) if it did not have a follower at the beginning of stage s, σ must have a follower p_i at the end of stage s. By choice of s, σ is uninitialized at every stage $t > s$, so p_i remains the follower for σ at all stages $t \geq s$.

(ii): By Lemma 4.4(i),(ii), there is a stage s such that for all $t \geq s$, λ_t does not have higher priority than $\sigma^\frown\langle w\rangle$ and $\sigma^\frown\langle w\rangle$ is uninitialized at stage t. By (i), there is a number p_i which is the follower of σ at all stages $t \geq s$. By Lemma 4.5, there are infinitely many $t \geq s$ such that $\lambda_t \supseteq \sigma^\frown\langle w\rangle$ and Case 2 of the construction is followed at stage t. For any such t, if $\Phi^t_{h(\alpha_i)}(\oplus\widetilde{I}^t_{\alpha_i,n,a}; p_i)\downarrow = D^t_{\alpha_i,n,a,h_{\alpha_i}}(p_i)$ and $\Phi^t_{h(\alpha_i)}(\oplus\widetilde{I}^t_{\alpha_i,n,a}; p_i)\downarrow = 0$, then σ would require, and hence would receive attention for (3.2) at stage t, and we would set $\sigma^\frown\langle c\rangle \subseteq \lambda_t$ and initialize $\sigma^\frown\langle w\rangle$ at stage t, contrary to the choice of s. As $D_{\alpha_i,n,a,h_{\alpha_i}}(p_i) = 0$, it follows that $\Phi^t_{h(\alpha_i)}(\oplus\widetilde{I}^t_{\alpha_i,n,a}; p_i) \neq D_{\alpha_i,n,a,h_{\alpha_i}}(p_i)$ for infinitely many t, and so (ii) follows.

(iii): By Lemma 4.4(i),(ii), there will be a stage s_0 such that for all $t \geq s_0$, λ_t does not have higher priority than $\sigma^\frown\langle r\rangle$ and $\sigma^\frown\langle r\rangle$ is uninitialized at stage t. By (i), there is a number p_i which is the follower of σ at all stages $t \geq s_0$. Note that if $\sigma^\frown\langle r\rangle \subset \Lambda$, then by the construction, there must be infinitely many stages t such that $\sigma^\frown\langle c\rangle \subseteq \lambda_t$. Suppose that (iii) fails, in order to obtain a contradiction. Then there must be a stage $t \geq s_0$ and a number u_i^t such that $\sigma^\frown\langle c\rangle \subseteq \lambda_t$, $\Phi_{h(\alpha_i)}(\oplus\widetilde{I}^t_{\alpha_i,n,a} \upharpoonright u_i^t + 1; p_i)\downarrow$, and $\oplus\widetilde{I}^v_{\alpha_i,n,a} \upharpoonright u_i^t + 1 = \oplus\widetilde{I}^t_{\alpha_i,n,a} \upharpoonright u_i^t + 1$ for all $v \geq t$. But then by choice of s_0, we will never initialize $\sigma^\frown\langle c\rangle$ at any stage $v \geq t$, so $\sigma^\frown\langle r\rangle \not\subset \Lambda$, contrary to hypothesis. $\square$

The next lemma will allow us to show that the requirement assigned to σ is satisfied whenever the true path extends the satisfaction outcome for σ.

Lemma 4.9. *Suppose that $\tau = \eta^\frown\langle z\rangle \in T$ is a node of the implementation phase of a copy M of a module for requirement $R_{\gamma(\eta),n,a,h^+_{\gamma(\eta)}}$. Fix i such that $\gamma(\eta) = \alpha_i$, and the node $\sigma \subset \eta$ of the design phase of M for which $\gamma(\sigma) = \alpha_i = \gamma(\eta)$. Suppose that $s < \widehat{s}$, $\lambda_s \supseteq \sigma^\frown\langle c\rangle$ but $\sigma^\frown\langle c\rangle$ is initialized at stage $s-1$, λ_t does not have higher priority than $\sigma^\frown\langle c\rangle$ for any t such that $s \leq t \leq \widehat{s}$, η is uninitialized at the beginning of stage $\widehat{s}$ and τ is uninitialized at the end of stage $\widehat{s}$. Then $\sigma_i^s \subseteq \oplus\widetilde{I}^{\widehat{s}}_{\alpha_i,n,a}$, $\Phi^{\widehat{s}}_{h(\alpha_i)}(\sigma_i^s; p_i)\downarrow = 0$ and $p_i \in D_{\alpha_i,n,a,h_{\alpha_i}}$.*

Proof. As λ_t does not have higher priority than $\sigma^\frown\langle c\rangle$ for any t such that $s \leq t \leq \widehat{s}$, σ_i^s is not canceled at any such stage, so $\sigma_i^t = \sigma_i^s$. As η is uninitialized at the beginning of stage $\widehat{s}$ and τ is uninitialized at the end of stage $\widehat{s}$, we must have $\sigma_i^s \subseteq \oplus\widetilde{I}^{\widehat{s}}_{\alpha_i,n,a}$, else (3.5) would apply to η and we would follow Case 1 of the construction either for η or a node of higher priority than η; in either case, $\tau = \eta^\frown\langle z\rangle$ would be initialized at the end of stage $\widehat{s}$. We now note that σ receives attention at stage s, $\Phi^s_{h(\alpha_i)}(\sigma_i^s; p_i) \downarrow= 0$, and $\sigma_i^s \subseteq \oplus\widetilde{I}^{\widehat{s}}_{\alpha_i,n,a}$. Hence $\Phi^{\widehat{s}}_{h(\alpha_i)}(\oplus\widetilde{I}^{\widehat{s}}_{\alpha_i,n,a}; p_i) \downarrow= 0$. Now as $\tau = \eta^\frown\langle z\rangle$, the construction must have acted through (3.4) to place $p_i \in D^{t+1}_{\alpha_i,n,a,h_{\alpha_i}}$ at the first stage $t \geq s$ at which τ is uninitialized at the end of stage t. □

We now define, for each integer n and $a \in B$, a subtree $T'_{n,a}$ of T_n and the set $G_{n,a}$ of immediate extensions of nodes of $T'_{n,a}$ within T_n. We will simultaneously define a labeling g of $T'_{n,a}$. These sets will be used in the following way. There are two components to each requirement, a diagonalization component and a permission component requiring computations of the set D_{α,n,a,h_α}. Within an individual module M which is a copy of the enhanced tree $\widetilde{T}^h_n$ on T, our strategy will be to respond to moves made by the opponent, without trying to force satisfaction of the requirement at a particular node. $T'_{n,a}$ will be the longest initial portion of T_n such that for nodes in that set, the opponent failed to satisfy some diagonalization requirement, and $G_{n,a}$ will be the set of minimal nodes, within T_n, which are not on $T'_{n,a}$. Thus full diagonalization succeeds for nodes in $G_{n,a}$. The construction is also arranged so that for nodes $\alpha \in G_{n,a}$, the permission component of the requirement will also succeed for any full labeling h of T_n such that $h \supseteq g_\alpha$. Thus if we can show that $G_{n,a} \neq \emptyset$, then we will know that we have satisfied all requirements $R_{\alpha,n,a,h^+_\alpha}$ for any labeling of T_n which extends g_α.

We will next have to show that $G_{n,a} \neq \emptyset$. We will show that if this fails, then g is a full labeling of T_n, and when a copy M of this module appears along Λ, it is impossible to follow any terminal outcome of M, so $|\Lambda| < \infty$ contrary to what we have shown.

We will then define a binary tree consisting of all initial segments of characteristic functions of the indices of the c.e. sets in the ideal coded by elements of $G_{n,a}$. As $G_{n,a}$ is non-empty for each n, this will be an infinite binary tree, so will have an infinite branch by König's Lemma. All requirements will be satisfied for this choice of the ideal; for ideals with indices in $\{0, \ldots, n\}$, the witness to the satisfaction may not come from $G_{n,a}$, but will come from $G_{m,a}$ for some $m \geq n$.

Definition 4.10. We define a subtree $T'_{n,a}$ of T_n for each integer n and each $a \in B$, and a function g with domain $T'_{n,a}$ by induction on $|\nu|$ for $\nu \in T_n$. Suppose that we have completed the induction for all $\tau \subset \nu$. If

$$\{k : \Phi_k(\oplus\widetilde{I}_{\nu,n,a}) \neq D_{\nu,n,a,g_\nu}\} = \omega, \tag{4.1}$$

fails to hold, then we place ν on $T'_{n,a}$ and define $g(\nu)$ to be the smallest k for which $\Phi_k(\oplus\widetilde{I}_{\nu,n,a}) = D_{\nu,n,a,g_\nu}$; otherwise, neither ν nor any of its extensions are placed on $T'_{n,a}$, and we place $\nu \in G_{n,a}$.

Lemma 4.11. *For all integers n and for all coatoms $a \in B$, $G_{n,a} \neq \emptyset$.*

Proof. We assume that $G_{n,a} = \emptyset$ and derive a contradiction. Under this assumption, g labels all of T_n, so the tree T^g_n is well-defined, as its enhanced counterpart $\widetilde{T}^g_n$, and by Lemma 4.4(iv), there will be a copy M, on T, of a module whose underlying tree is $\widetilde{T}^g_n$, whose initial node is $\delta \subset \Lambda$, and which is assigned requirement $R_{n,a,g}$. Furthermore, the definition of $T'_{n,a}$ ensures that for every $\nu \in T_n$, $\Phi_{g(\nu)}(\oplus \widetilde{I}_{\nu,n,a}) = D_{\nu,n,a,g_\nu}$. Now as each module is finite, there will be a terminal node $\delta^\frown\xi \subset \Lambda$ of M. Let $\alpha = \xi^-$, $\alpha_i = \gamma(\alpha)$, and $\Phi = \Phi_{g(\alpha_i)}$. By the definition of g,

$$\Phi(\oplus \widetilde{I}_{\alpha_i,n,a}) \text{ is total} \tag{4.2}$$

and

$$\Phi(\oplus \widetilde{I}_{\alpha_i,n,a}) = D_{\alpha_i,n,a,g_{\alpha_i}}. \tag{4.3}$$

As $\delta^\frown\xi$ is a terminal node of M, there is an $o \in \{r, w, z\}$ such that $\xi = \alpha^\frown\langle o\rangle$. By Lemma 4.8, (4.2) and (4.3), $o \notin \{r, w\}$. But by Lemma 4.9 and (4.3), $o \neq z$ yielding the desired contradiction. □

We now show that each string in $G_{n,a}$ gives rise to requirements which are satisfied. The satisfaction of a requirement has two components; diagonalization and the computation of a set D from all sets in a filter. For requirements corresponding to nodes in $G_{n,a}$, diagonalization is easily seen to be satisfied just by the definition of $G_{n,a}$. (Hence it is never necessary to find a node $\subset \Lambda$ which witnesses the satisfaction of the diagonalization requirement.) The proof of the lemma concentrates on showing that D is computable from all sets in a specified filter.

Lemma 4.12. *Fix $n < \omega$ and let a be a coatom of B. Suppose that $\nu \in G_{n,a}$. Then for all computable partial functionals Φ, $\Phi(\oplus \widetilde{I}_{\nu,n,a}) \neq D_{\nu,n,a,g_\nu}$ and for all $V \in \widetilde{F}_{\nu,n,a}$, $D_{\nu,n,a,g_\nu} \leq_T V$.*

Proof. Let $F = \widetilde{F}_{\nu,n,a}$, $I = \widetilde{I}_{\nu,n,a}$ and $D = D_{\nu,n,a,g_\nu}$.

By the definition of $G_{n,a}$, for all computable partial functionals Φ, $\Phi(\oplus I) \neq D$, and for all $\rho \subset \nu$, $\Phi_{g(\rho)}(\oplus \widetilde{I}_{\rho,n,a}) = D_{\rho,n,a,g_\rho}$. It remains to show that $D \leq_T V$ whenever $V \in F$. Let $\mathcal{M}$ be the set of all modules for the construction for requirements of the form $R_{n,a,h}$ where $h \supseteq g_\nu$. (Note that if $h \supseteq g_\nu$, then $\rho \subset \nu$ implies $h_\rho = g_\rho$.)

Any number p which might enter D is either appointed as a follower for a module $M \in \mathcal{M}$ and is assigned to the tree of strategies before stage p, or cannot enter D. Suppose that p is such a follower. If $p \in D^p$, then $p \in D$. So suppose that $p \notin D^p$. We consider the two possible cases.

First assume that $V = C_b$ for some $b \in B$ such that $b \not\leq a$. Then $b \geq a'$, so a permission witness q for b is appointed when p is appointed, and will enter its target set if and only if p enters its target set (and both will enter at the same stage). Hence $p \in D$ iff $q \in V$.

Now suppose that $V = W_m \in F$. Fix the unique nodes η of the implementation phase of M and α of the design phase of M such that $\gamma(\eta) = \gamma(\alpha) = \nu$, and the unique node β of M at which the implementation phase begins, noting that $\alpha \subset \beta \subseteq \eta$. As $V = W_m \in F$, there must be nodes σ and ξ of M such that $\beta \subseteq \xi$, $\sigma \subset \alpha$, $\gamma(\xi) = \gamma(\sigma)$, and $\xi^\frown\langle m\rangle \subseteq \eta$. Fix i such that $\gamma(\sigma) = \alpha_i$.

Fix the smallest stage $s \geq p$ such that $W^s_m \restriction p + 1 = W_m \restriction p + 1$. We assume that $p \notin D^s$ and p is still a follower, else we are done. We now claim that there is a smallest stage $t \geq s$ at which one of the following occurs:

(i) $p \in D^t$.

(ii) α is initialized at stage t.
(iii) β is initialized at stage t.
(iv) There are ρ and $o \neq z$ such that $\beta \subseteq \rho \subset \eta$, $\lambda_t \supseteq \rho^\frown\langle o\rangle$ and $\rho^\frown\langle o\rangle \nsubseteq \eta$.

To see the claim, assume that (ii)–(iv) fail for all $t \geq s$. As $\sigma \subset \alpha$ and (ii) fails for all $t \geq s$, u_i^s, p_i^s and σ_i^s are defined, and for all $t \geq s$, $u_i^t = u_i^s$, $p_i^t = p_i^s$ and $\sigma_i^t = \sigma_i^s$. Now the failure of (iii) implies that there is a longest ρ such that $\beta \subseteq \rho \subseteq \eta$ which is uninitialized at some stage $s_0 \geq s$. It cannot be the case that $\rho \subset \eta$ and $\rho^\frown\langle z\rangle \subseteq \lambda_t$ for infinitely many $t \geq s_0$, else by Lemma 4.9, $\Phi_{g(\gamma(\rho))}(\oplus\widetilde{I}_{\gamma(\rho),n,a}) \neq D_{\gamma(\rho),n,a,h_{\gamma(\rho)}}$, contrary to the choice of $h \supseteq g_\nu$. Hence by Remark 4.1, there will be a $t \geq s$ and an outcome $o \neq z$ such that $\rho^\frown\langle o\rangle \subseteq \lambda_t$. Now the failure of (iv) and the maximality of ρ imply that $\rho = \eta$. As $p \notin D^s$, η must have been initialized at stage s, so t is the first stage $\geq s$ at which η is uninitialized. We now see that p cannot have been released before stage t, so (3.5) cannot hold. Thus (3.4) must hold and we place $p \in D^t$, and so (i) holds.

We now search for the first $t \geq s$ at which one of the clauses (i)–(iv) holds. Note that t can be effectively determined from m. If (i) holds, then $p \in D$. If (ii) holds, then p is canceled and never again reappointed as a follower, so $p \notin D$. If (iii) holds but (ii) fails to hold, then by Lemma 4.7 and as $\sigma \subset \alpha$ implies $u_i^s < p$, $p \notin D$. Finally, if (iv) holds but (ii) and (iii) fail to hold, then by Lemma 4.2(iii), $p \notin D$. We have thus shown that $D \leq_T V = W_m$. □

Definition 4.13. For each $\nu \in T_n$, let χ_ν be the characteristic function of the subset of $\{0, \ldots, n\}$ specified by the range of ν. Define the binary tree X_a (ordered by inclusion) by placing

$$\delta \in X_a \;\leftrightarrow\; \exists n \exists \nu \in G_{n,a}(\delta \subseteq \chi_\nu).$$

By Lemma 4.11 $G_{n,a} \neq \emptyset$ for all n and a. As all elements χ_ν for $\nu \in G_{n,a}$ have length $n+1$, it follows that X_a is an infinite binary tree; hence by König's Lemma, X_a has an infinite branch Γ_a.

For $a \in B$, we will define subsets of the representations of computably enumerable sets provided by the standard enumeration, together with the sets $\{C_b : b \in B\}$. (Thus the same computably enumerable set may appear infinitely often with a different name each time; the name will either have the form W_i or C_b.) Define $\widehat{I}_a = \{C_b : b \in B\} \cup \{W_i : i < \omega\}$ if $a = 1$,

$$\widehat{I}_a = \{C_b : b \leq a\} \cup \{W_m : \Gamma_a(m) = 0\}$$

if a is a coatom of B, and

$$\widehat{I}_a = \bigcap \{I_b : b \geq a \;\&\; b \text{ is a coatom of } B\}$$

otherwise. We will show that these sets are degree-invariant. For each $a \in B$, define I_a to be the set of degrees of elements of $\widehat{I}_a$. We also show that the map f described in (2.7) induces a degree-preserving pseudolattice homomorphism.

Lemma 4.14. *The sets $\widehat{I}_a$ are degree-invariant and f is a homomorphism of $\mathcal{R}_p$ onto $\mathcal{B}$.*

Proof. It follows from Lemma 4.12 that if X and Y are names of computably enumerable sets as described above, $X \leq_T Y$, and $Y \in \widehat{I}_a$ then $X \in \widehat{I}_a$. Hence the sets $\widehat{I}_a$ are degree-invariant.

We now verify the hypotheses of Lemma 2.6. (2.1) is immediate. (2.2) is vacuous, as all meet irreducible elements of a boolean algebra which are not 1 are coatoms, and no two coatoms are comparable. (2.4) follows from the definitions of $\widehat{I}_a$ and I_a. Lemma 4.12 and Remark 3.4 ensure the satisfaction of (2.5).

Fix $a, b \in B$ such that $a \not\geq b$. Then $C_b \in \widehat{I}_b - \widehat{I}_a$, whence (2.3) follows from the degree-invariance of these sets.

It now follows from Lemma 2.6 that f is a pseudolattice homomorphism of $\mathcal{R}_p$ onto $\mathcal{B}$. $\square$

We have thus proved:

Theorem 4.15. *Let $\mathcal{B}$ be a finite boolean algebra. Then there is a pseudolattice homomorphism from $\mathcal{R}_p$ onto $\mathcal{B}$.*

The proof presented above differs from that in Calhoun [2]. While the underlying basic module is similar, Calhoun used the Recursion Theorem to avoid describing the inner workings of the construction, while we have tried to present a detailed description which can be implemented as an algorithm.

The passage from boolean algebras to distributive lattices requires us to take into account comparable meet-irreducible elements, so the verification of (2.2) is no longer trivial. We will take additional steps to satisfy this condition, which will suffice for a certain class of distributive lattices. The proof is presented in the next section.

5. The distributive lattice case

We now turn to the construction of a homomorphism of $\mathcal{R}_p$ onto a biorderable finite distributive lattice $\mathcal{L}$ with universe L. Let L_{MI} be the set of meet-irreducible elements of L which are $\neq 1$. Finite distributive lattices are not uniquely generated by taking meets of coatoms, but are uniquely generated by taking meets of irredundant sets of meet-irreducible elements (an *irredundant* set is a set I such that $\wedge I \neq \wedge J$ for any $J \subset I$). The presence of comparable elements in L_{MI} deprives us of the ability to define the ideals I_a independently of each other. And we need ideals to *cohere*, i.e., we require that if $a, b \in L_{MI}$ and $a < b$, then $I_a \subset I_b$. To see the problem, consider the linearly ordered set with elements $0 < a < 1$. If we try to define the ideals I_0 and I_a independently, there seems to be no way to prevent specifying that $W_0 \in I_0$ when we carry out the construction for the meet-irreducible element 0 and specifying that $W_0 \notin I_a$ when we carry out the construction for the meet-irreducible element a; as we must have $I_0 \subset I_a$, the ideals do not cohere. The problem can be resolved by coordinating the actions of modules M_0 for 0 and M_a for a through a modification of the procedure described in Definition 3.5, by merging the two modules into a hybrid module. We first implement the design phase of M_a, then the design phase of M_0, then the implementation phase of M_a, and finally the implementation phase of M_0. The permissions by sets W_i observed while implementing M_a can then also be used by M_0 as permissions; and the permissions by the sets C_b which result from action during the implementation phase for M_a will not injure the computation that M_0 is trying to preserve.

The above strategy will succeed when the maximal chains of L_{MI} partition L_{MI}, as we can then carry out this procedure independently for each chain. When the maximal chains intersect, however, there is no way to avoid injury to computations which need to be preserved when sets C_a permit through an implementation of this strategy. Thus we must allow the modules to act somewhat independently of one another as was the case with boolean algebras, yet modify the construction in a way that forces the combined action to cohere. We list the basic ideas to be employed, and demonstrate their use in the next example.

- We will handle the design phases and implementation phases for modules for several elements of L_{MI} in a single hybrid module.
- All design phases will be completed before any implementation phase begins.
- If $a < b \in L_{MI}$, then the design phase for the module M_b for b will be completed before the design phase for the module M_a for a begins, and the implementation phase for M_b will be completed before the implementation phase for M_a begins. Thus all permissions within M_b can be used as permissions by M_a.
- If a, b are incomparable elements of L_{MI} (we write $a|b$) then the relative ordering of M_a and M_b in the design phase is the reverse of that in the implementation phase. More specifically, if the implementation phase for M_a precedes that for M_b, then the design phase for M_b precedes that for M_a. This prevents injury to M_b before it is implemented, as the action taken to implement M_a uses numbers larger than those restrained by M_b.

Example 5.1. We describe a procedure for determining a coherent assignment of W_0 and W_1 to ideals corresponding to the meet-irreducible elements in the five-element distributive lattice generated by meet-irreducibles a, b, 0 and 1 with $a \vee b = 1$ and $a \wedge b > 0$. The procedure uses hybrid modules which have the above properties, but for which some outcomes of the individual models may have been removed.

We linearly order L_{MI} so that a precedes b which, in turn, precedes 0. We will consider all possible hybrid modules. Suppose that we have outcomes for a and b whose corresponding requirements have been satisfied, and consider the hybrid module with these outcomes removed. We first carry out the design phase of a $\langle 1, a\rangle$-module M_a with corresponding outcomes removed, then the design phase of a $\langle 1, b\rangle$-module M_b with corresponding outcomes removed, and then the design phase of a $\langle 1, 0\rangle$-module M_0. (The idea is that when a requirement is satisfied for a given node σ of the tree giving rise to the module (so the satisfaction is for a fixed oracle and element of L_{MI}), then we need only consider hybrid modules obtained by removing the nodes which correspond to the oracle and element $a \in L_{MI}$ for which the requirement is satisfied; the removal covers both the design and implementation phases, and when a node is removed, its extensions on that individual model are also removed. Attempts for this requirement are no longer necessary as the requirement has been satisfied, and the permissions encountered along the path through σ can be used by later nodes of the hybrid module working for $c \in L_{MI}$ with $c < a$.)

We first implement the middle module M_a. Suppose that M_a reaches an outcome which has been removed from the hybrid module. As the design phase for M_a is completed after that for M_b, all numbers which M_a targets for sets C_d are too large to injure any of the computations restrained by the design phase for M_b. Thus it is safe to now begin implementing M_b; we do so, and never again act for M_a, as the implementation of M_b will destroy our ability to use anything about M_a except its permissions. M_b now acts independently, and there is no interference between the action of either M_a or M_b with the operation of M_0. M_0 is the last to be implemented, and will cohere with the outcomes of the two preceding modules. While we have lost the correctness of M_a in the process, the corresponding requirement will have already been satisfied, so will provide the outcome needed for coherence when König's Lemma is applied.

Suppose that M_a has a final outcome which was not removed from the module. We will then have another hybrid module in which this new outcome is also removed. We have now collected another outcome of a $\langle 1, a\rangle$-module to be used when we apply König's Lemma. As there are only finitely many possible outcomes, we can continue in this way and eventually find a matched set of consistent outcomes.

Example 5.1 points the way to a definition of n-modules which is consistent with the needs of the homomorphism under construction. For each hybrid module, we will need to specify the order in which we carry out the components of the design phases of the various modules, the order in which we carry out the components of the implementation phases of the various modules (these orders may differ), and the outcomes previously encountered which cause us to pass to the module for the next element of L_{MI}. We will specify the properties that such orderings must have in order for our construction to succeed, and prove the homomorphism theorem for distributive lattices for which such orderings exists. At the end of this section, we will give some examples of finite distributive lattices which possess such orderings, and some which do not.

We will need two different orderings of L_{MI}; $<_d$ for the design phase and $<_p$ for the implementation phase. Suppose that $a_i \not\geq a_j$. Action for M_{a_i} will dictate the placement of numbers into C_{a_j}, while action for M_{a_j} will impose restraint on C_{a_j}. Hence in order to be able to implement M_{a_j} before an injury renders it incapable of satisfying a requirement, we must require either that $a_j \leq_d a_i$ so that the numbers M_{a_i} places into C_{a_j} are too large to injure any computation which M_{a_j} wishes to preserve, or that M_{a_j} is implemented first. In the latter case, if M_{a_j} provides us with a collection of permitted c.e. sets which we have previously encountered, then we do not need this module, so it is safe to let M_{a_i} act afterwards. We also note that if $a_j \leq a_i$, then M_{a_i} will never act to place numbers into C_{a_j}. Thus as noted earlier, we will require our orderings to have the properties introduced in the next definition.

Definition 5.2. We say that the finite distributive lattice $\mathcal{L}$ with universe L is *biorderable* if there are a pair of linear orderings $<_d$ and $<_p$ on L_{MI} which satisfy the following conditions for all $a, b \in L_{MI}$:

$$a < b \;\rightarrow\; b <_d a \;\&\; b <_p a. \tag{5.1}$$

$$a|b \;\rightarrow\; (a <_p b \;\leftrightarrow\; b <_d a). \tag{5.2}$$

We call the pair of orderings $\langle <_d, <_p\rangle$ a *biordering* of L_{MI}. We specify a biordering $\langle <_d, <_p\rangle$ of L_{MI}, and index the elements of L_{MI} in the order induced by $<_p$, i.e.,

$$a_0 <_p \cdots <_p a_m. \tag{5.3}$$

It is easy to see that for any subset S of L_{MI}, if we restrict any pair of orderings of L_{MI} having these properties to S, we obtain orderings of S which also have these properties.

In defining hybrid modules, we will need to talk about permission outcomes along a path, as these will have an impact on the passage between stages within the module. As we will be working with all elements of L_{MI} within the same hybrid module, the outcome will code not only the sets which have permitted, but also the element of the lattice whose requirement has been satisfied. Thus we will define a map α sending nodes σ of a tree, to elements $\alpha(\sigma)$ of L_{MI}.

Definition 5.3. Let T be a tree of sequences from some set. A *selection map* for T is a partial map α which sends elements of T to L_{MI}, whose domain is closed under inclusion, and which satisfies the condition

$$\sigma \subset \tau \;\&\; \alpha(\sigma) = a_i \;\&\; \alpha(\tau) = a_j \;\rightarrow i \leq j.$$

Given $\sigma \in T$, α_σ will denote $\alpha \restriction \sigma$ and α_σ^+ will denote $\alpha_\sigma \cup \{\langle \sigma, \alpha(\sigma)\rangle\}$.

Basic modules will be replaced with hybrid n-modules, which will have both enhanced and unenhanced versions. These modules will depend on a set $\mathcal{O}$ coding the outcomes to be removed from consideration by the module. $\mathcal{O}$ will be a collection of triples $\langle \sigma, \alpha_\sigma, a\rangle$ where $\sigma \in \omega^{<\omega}$, α is a selection map and $a \in L_{MI}$, and is called an *outcome set*.

Before defining the hybrid trees, we will need to specify a process which takes a string σ, strips away some of its outcomes, and compresses the remainder into a new string. (The new string will identify the sets which have already permitted and whose permissions can be used to satisfy the requirement currently assigned to σ. The index of the permitting set, which is what δ wishes to capture, is the outcome of a node τ, and depends on an element $a \in L_{MI}$.)

Definition 5.4. Fix $\sigma \in \omega^{<\omega}$, $a \in L_{MI}$, and a selection map $\alpha : \sigma \;\rightarrow\; L_{MI}$. We define

$$i \in \widehat{\delta}(\sigma, \alpha, a) \;\leftrightarrow\; \exists \rho \subset \sigma(\alpha(\rho) \geq a \;\&\; \rho^\frown\langle i\rangle \subseteq \sigma).$$

Let $\delta(\sigma, \alpha, a)$ be the string enumerating $\widehat{\delta}(\sigma, \alpha, a)$ in order of magnitude.

Remark 5.5. It follows easily from the preceding definition that if $\tau \subset \sigma$ then $\widehat{\delta}(\tau, \alpha_\tau, a) \subseteq \widehat{\delta}(\sigma, \alpha, a)$. Furthermore, if $\alpha(\rho) \not\geq a$ for all ρ such that $\tau \subseteq \rho \subset \sigma$, then $\widehat{\delta}(\tau, \alpha_\tau, a) = \widehat{\delta}(\sigma, \alpha, a)$.

Definition 5.6. Let $n < \omega$ and an outcome set $\mathcal{O}$ be given. We define the *hybrid n-tree* $T(n, \mathcal{O})$ by induction on the numbers $\leq m$, the number specified in (5.3). We will have a set of *initial nodes* at the beginning of each step of the induction, and a set of *terminal nodes* at the end of each step, as well as the portion of the selection map $\alpha = \alpha_{n,\mathcal{O}}$ defined by the end of Step $j-1$ whose domain consists of the non-terminal nodes on the portion of $T(n, \mathcal{O})$ which is currently defined. $\emptyset$ is the only initial node for Step 0.

Suppose that σ is an initial node for Step j, and let $\nu = \delta(\sigma, \alpha_\sigma, a_j)$. Let $\widetilde{\alpha}$ be the selection map whose domain is the set of strings which are compatible with σ and which is defined by $\widetilde{\alpha}(\tau) = \alpha(\tau)$ if $\tau \subset \sigma$ and $\widetilde{\alpha}(\tau) = a_j$ if $\tau \supseteq \sigma$. We place $\sigma^\frown\xi \in T(n, \mathcal{O})$ at Step j for every ξ satisfying the following conditions:

$$\nu^\frown\xi \text{ is a sequence of elements from } \{0, \ldots, n\} \text{ without repetitions,}$$
$$\langle \sigma^\frown\delta, \widetilde{\alpha}_{\sigma^\frown\delta}, a_j\rangle \notin \mathcal{O} \text{ for all } \delta \subseteq \xi;$$

and for all such ξ, we define $\alpha(\xi) = a_j$. For every string ξ of minimal length which satisfies the first of these conditions but fails to satisfy the second, $\sigma^\frown\xi$ is an initial node for Step $j+1$ if $j < m$, and a terminal node of $T(n, \mathcal{O})$ if $j = m$. In the latter case, we set $\alpha(\sigma^\frown\xi) = a_m$.

A *labeling* of $T(n,\mathcal{O})$ is a function $h : T(n,\mathcal{O}) \to \omega$. We let $T(n,\mathcal{O},h)$ denote the tree with label h, and define the selection map $\alpha_{n,\mathcal{O},h}$ for this tree to be identical with $\alpha_{n,\mathcal{O}}$.

Note that a hybrid n-tree is finite-branching, consists of strings from a finite set, and its selection function α is total and must satisfy the condition

$$(\sigma \subset \tau \ \&\ \sigma(|\sigma|-1) = \tau(|\tau|-1)) \ \to\ \alpha(\sigma^-) \neq \alpha(\tau^-).$$

Hence as L_{MI} is finite, each hybrid tree is finite.

Within a given hybrid n-tree $T(n,\mathcal{O})$, we will interleave the steps for different elements of L_{MI}. The priority ordering will follow the lexicographical ordering of the enhanced hybrid *n-tree* $\widetilde{T}(n,\mathcal{O})$ which is now defined.

Definition 5.7. Let $T(n,\mathcal{O})$ be a hybrid n-tree with selection map $\alpha_{n,\mathcal{O}}$. The *enhanced hybrid n-tree* $\widetilde{T}(n,\mathcal{O})$ is defined from $T(n,\mathcal{O})$ in a manner analogous to that in which $\widetilde{T}^h_n$ was defined from T^h_n in Definition 3.3, (but ignoring the labeling provided by h). We also defined a map $\gamma(\sigma)$ in Definition 3.3, and that definition required us to specify an ordering of the nodes of $T(n,\mathcal{O})$ as $\alpha_0,\dots,\alpha_{k-1}$. Such a map $\gamma_{n,\mathcal{O}}$ is again defined, using the ordering which is uniquely determined by the following conditions. Suppose that $\sigma,\tau \in T(n,\mathcal{O})$. Fix a and b such that $\alpha_{n,\mathcal{O}}(\sigma) = a$ and $\alpha_{n,\mathcal{O}}(\tau) = b$. Then we say that $\sigma \prec \tau$ if one of the following conditions holds:

$$a <_d b. \tag{5.4}$$

$$a = b \ \&\ \sigma <_\ell \tau. \tag{5.5}$$

This tree comes equipped with a selection map $\widetilde{\alpha}_{n,\mathcal{O}}$ sending the non-terminal nodes of $\widetilde{T}(n,\mathcal{O})$ to elements of L_{MI} defined by $\widetilde{\alpha}_{n,\mathcal{O}}(\sigma) = \alpha_{n,\mathcal{O}}(\gamma_{n,\mathcal{O}}(\sigma))$. The labeled tree $\widetilde{T}(n,\mathcal{O},\widetilde{h})$ is defined from $T(n,\mathcal{O},h)$ by setting $\widetilde{h}(\tau) = h(\gamma_{n,\mathcal{O}}(\tau))$ whenever τ is a non-terminal node of $\widetilde{T}(n,\mathcal{O},h)$, and its selection map is defined by $\widetilde{\alpha}_{n,\mathcal{O},h} = \widetilde{\alpha}_{n,\mathcal{O}}$. We noted in Section 4 that the process for enhancing a finite tree produces a finite tree. Thus each enhanced hybrid tree is finite.

Fix $\sigma \in T(n,\mathcal{O},h)$ and the selection map $\alpha = \alpha_{n,\mathcal{O},h}$. Recall that h_σ is the restriction of h to the nodes $\subset \sigma$ and that h^+_σ is the restriction of h to the nodes $\subseteq \sigma$. Below, we treat two sets with different names as different elements, even though they both may name the same subset of ω. (Thus the same computably enumerable set may appear infinitely often with a different name each time; the name will either have the form W_i or C_b.) We define sets

$$F_{\sigma,n,\alpha} = \{W_i : i \le n \ \&\ i \in \widehat{\delta}(\sigma,\alpha_\sigma,\alpha(\sigma))\},\ I_{\sigma,n,\alpha} = \{W_i : i \le n\} - F_{\sigma,n,\alpha},$$

$$\widetilde{F}_{\sigma,n,\alpha} = \{C_b : b \nleq \alpha(\sigma)\} \cup F_{\sigma,n,\alpha} \text{ and } \widetilde{I}_{\sigma,n,\alpha} = \{C_b : b \le \alpha(\sigma)\} \cup I_{\sigma,n,\alpha}.$$

The requirement for σ and α will be

$$R_{\sigma,n,\alpha,h^+_\sigma} :\ \Phi_{h^+_\sigma(\sigma)}(\oplus \widetilde{I}_{\sigma,n,\alpha}) \neq D_{\sigma,n,\alpha_\sigma,h_\sigma} \&\ \forall V \in \widetilde{F}_{\sigma,n,\alpha}(D_{\sigma,n,\alpha_\sigma,h_\sigma} \le_T V).$$

We assign the requirement $R_{\sigma,n,\alpha,h^+_\sigma}$ to σ, and note that if $\sigma = \tau$ as strings with $\tau \in T(n,\widehat{\mathcal{O}},g)$, $\alpha_\sigma = (\alpha_{n,\widehat{\mathcal{O}},g})_\tau$ and $h^+_\sigma = g^+_\tau$, then $R_{\sigma,n,\alpha,h^+_\sigma}$ and $R_{\tau,n,\alpha_{n,\widehat{\mathcal{O}},g},g^+_\tau}$ are identical.

Requirements are associated with labeled hybrid trees, and we let $R_{n,\mathcal{O},h}$ denote the requirement associated with $T(n,\mathcal{O},h)$. We fix an effective *priority listing* $\langle R_j : j < \omega\rangle$ of all requirements $R_{n,\mathcal{O},h}$ which arise in this way.

We will define the *hybrid n-module* $M_{n,\mathcal{O},h}$ as in Definition 3.5 modified to use the trees of this section.

Definition 5.8. A *hybrid n-module* $M_{n,\mathcal{O},h}$ is used to determine the ideals in which $W_0,\dots,W_n$ are placed. It is defined as in Definition 3.5, replacing the tree $\widetilde{T}^h_n$ with $\widetilde{T}(n,\mathcal{O},h)$ and setting $\alpha = \alpha_{n,\mathcal{O},h}$, the selection map for $T(n,\mathcal{O},h)$, and $\gamma = \gamma_{n,\mathcal{O},h}$. The ordering for the design phase is given by the ordering $\prec$ introduced in Definition 5.7. The ideal $\widetilde{I}_{\alpha,\alpha_i,n,a}$ is replaced by $\widetilde{I}_{\gamma(\sigma),n,\alpha}$ which we have just defined, and we set $\alpha_i = \gamma(\sigma)$, as determined by this tree. (The element $a \in L_{MI}$ is provided as $\widetilde{\alpha}_{n,\mathcal{O},h}(\sigma)$.)

Our tree of strategies T is a level 2 or $\mathbf{0}''$ tree. This tree will be built by gluing copies of enhanced hybrid n-trees to $\sigma = \emptyset$ to terminal nodes σ of earlier enhanced hybrid n-trees, always choosing a tree for the first R_i which has not yet been assigned to a predecessor of σ. This tree comes equipped with a priority ordering $<_\ell$ which is induced by its lexicographical ordering. A *path* through the tree will be an infinite sequence coded by the outcomes of the nodes of the modules, and the priority ordering is defined for paths as well.

Remark 5.9. We will want a selection function α for T and a map γ identifying nodes of T with nodes of hybrid trees. As each node $\sigma \in T$ is placed there as the counterpart of a unique node η of some $\widetilde{T}(n,\mathcal{O},h)$, the values of these maps on σ are taken to be the same as the values of the corresponding maps on η. The requirement associated with σ is now seen to be $R_{\gamma(\sigma),n,\alpha_{n,\mathcal{O},h},h^+_{\gamma(\sigma)}}$.

The construction and the definition of the *true path* Λ are almost identical to those in Section 4, so we do not repeat them here.

We now note that the proofs of Remark 4.1 and Lemma 4.2 require only the notational changes to identify the trees.

The proof of Lemma 4.3 handles all cases except one, the analysis required within an enhanced hybrid n-module. The lemma is restated below, and a proof is given which covers the new situation.

Lemma 5.10. *Let $\sigma \in T$ be a node in a copy M of the design phase of a hybrid module $M(n,\mathcal{O},h)$ with selection map $\widetilde{\alpha}_{n,\mathcal{O},h}$, and let the corresponding requirement be $R_{\gamma(\sigma),n,\widetilde{\alpha}_{n,\mathcal{O},h},h^+_{\gamma(\sigma)}}$. Fix i such that $\gamma(\sigma) = \alpha_i$, and fix ξ in the implementation phase of M such that $\gamma(\xi) = \gamma(\sigma)$. Suppose that we have stages $s < v$ such that $\sigma^\frown\langle c\rangle$ is initialized at the end of stage $s-1$ but $\sigma^\frown\langle c\rangle \subseteq \lambda_s$, we do not act to initialize σ or cancel σ^s_i at any stage t such that $s \le t \le v$, and that ξ receives attention at stage v through (3.5). Then for all $d \le \widetilde{\alpha}(\xi)$, $C^v_d \restriction u^s_i + 1 = C^s_d \restriction u^s_i + 1$.*

Proof. Suppose that the lemma fails, in order to obtain a contradiction. Then there is a $d \le \widetilde{\alpha}(\xi)$ and a number $q \in C^v_d - C^s_d$ such that $q \le u^s_i$. Let q be placed in C^{t+1}_d as part of the action for node ρ. This can only happen if $s \le t < v$ and ρ receives attention at stage t through (3.4). We note that ρ must be a node of the implementation phase of a module in order to receive attention through (3.4) while σ is a node of the design phase of its module, so $\rho \ne \sigma$; and if $\rho \subset \sigma$, then ρ and σ

must be in different modules. As $s \leq t < v$, we now see that ρ cannot have higher priority than σ else we would cancel σ_i^s during stage t, contrary to hypothesis.

We cannot have $\rho \supseteq \sigma^\frown\langle w\rangle$ and ρ cannot have lower priority than $\sigma^\frown\langle w\rangle$, as all such nodes are initialized at stage s and any numbers they later cause to be placed into sets are $> u_i^s$. We cannot have $\rho \supseteq \sigma^\frown\langle r\rangle$, as then we would act to cancel σ_i^s at stage t. It remains to consider the case in which $\rho \supseteq \sigma^\frown\langle c\rangle$. If ρ is not a node of M, then the node appointing q is initialized at the beginning of stage s so the numbers it appoints (q in particular) are $> u_i^s$.

Suppose that ρ is a node of M. Then as ρ places an element into a set, it must be the case that ρ is a node of the implementation phase of M. Let $\beta \subseteq \xi$ be the node of M at which the implementation phase begins. Then $\sigma \subset \beta$. Finally, let $\nu \subset \beta$ be the node of the design phase of M for which $\gamma(\nu) = \gamma(\rho)$. As σ_i^s is not initialized at any stage t such that $s \leq t \leq v$ and $q \leq u_i^s$, q must have been appointed before or at stage s, so $\nu \subseteq \sigma$.

Fix the smallest $s_0 \geq s$ such that β is uninitialized at all t such that $s_0 \leq t \leq v$ but β is initialized at stage $s_0 - 1$. As $\sigma^\frown\langle c\rangle \subseteq \beta$ is initialized at stage $s-1$, s_0 must exist. We must have $C_d^{s_0} \restriction u_i^s + 1 = C_d^s \restriction u_i^s + 1$, else σ would require attention through (3.3) at stage s_0, so some $\tau \subseteq \sigma$ would receive attention thereby either initializing σ (if $\tau \neq \sigma$) or canceling σ_i^s (if $\tau = \sigma$), contrary to hypothesis. Hence by the counterpart of Lemma 4.2 and the construction, v will be the least stage after s_0 at which ξ receives attention through (3.5), and we must have $\rho \subseteq \xi$. Now as $\rho \subseteq \xi$, $\nu \subseteq \sigma$, $\gamma(\rho) = \gamma(\nu)$ and $\gamma(\xi) = \gamma(\sigma)$, it follows from (5.1), (5.2), (5.4) and (5.5) that $\widetilde{\alpha}(\rho) \geq \widetilde{\alpha}(\xi)$. By the construction, action by ρ will not place any numbers into C_d if $d \leq \widetilde{\alpha}(\rho)$, so $d \nleq \widetilde{\alpha}(\xi) = \widetilde{\alpha}(\sigma)$, yielding a contradiction. □

We note that many of the lemmas of Section 4 now carry over unchanged.

Remark 5.11. The statements of Lemmas 4.4–4.9 and their proofs apply, as they stand, to the above construction, replacing the application of Lemma 4.3 in the proofs of Lemmas 4.4–4.9 with Lemma 5.10. The only changes required are the notational changes needed to refer to requirements and ideals.

The following definition will be used below.

Definition 5.12. Let $\nu \in \{0, \ldots, n\}^{<\omega}$ and maps $\widehat{\alpha}$ and g with domain ν be given. We say that ν *fails explicit diagonalization for* $\langle n, \widehat{\alpha}, g\rangle$ *with witness* $\langle j, k\rangle$ if for every copy M of a module $M(n, \mathcal{O}, h)$ lying on T and every $\rho \in M$, if $\gamma(\rho) = \nu$, $(\alpha_{n,\mathcal{O},h})_{\gamma(\rho)} = \widehat{\alpha}_\nu$, $(h_{n,\mathcal{O},h})_{\gamma(\rho)} = g_\nu$, $\alpha_{n,\mathcal{O},j}(\gamma(\rho)) = a_j$ and $h_{n,\mathcal{O},h}(\gamma(\nu)) = k$, then either

$$\rho^\frown\langle o\rangle \subset \Lambda \text{ for some } o \in \{w, r\}$$

or

$$\rho^\frown\langle z\rangle \text{ is uninitialized at cofinitely many stages.}$$

We now define $T'(n, j)$, $G(n, j)$, $\mathcal{O}(n, j)$, $\alpha'_{n,j}$ and $g_{n,j}$ by induction on $j \leq m$. The definitions are different from those of Section 4, but the properties of the sets and functions defined are similar.

Definition 5.13. We define $T'(n, j)$, $G(n, j)$, $\mathcal{O}(n, j)$, $\alpha'_{n,j}$ and $g_{n,j}$ by induction on $j \leq m$. We begin by setting $\mathcal{O}(n, 0) = \emptyset$. Fix j, and suppose that $\mathcal{O}(n, j)$ has been defined. We now proceed by induction on $|\nu|$ for $\nu \in T(n, \mathcal{O}(n, j))$. Suppose that we have completed the induction for all $\tau \subset \nu$, and have thus defined $g_{n,j}(\tau)$ and

$\alpha'_{n,j}(\tau)$ for all $\tau \subset \nu$. If there is a $k < \omega$ such that ν fails explicit diagonalization for $\langle n, (\alpha'_{n,j})_\nu, (g_{n,j})_\nu \rangle$ with witness $\langle j, k \rangle$, place $\nu \in T'(n,j)$, define $g_{n,j}(\nu)$ to be the smallest such k, and $\alpha'_{n,j}(\nu) = a_j$; otherwise, neither ν nor any of its extensions, is placed on $T'(n,j)$, we place $\nu \in G(n,j)$, and we place $\langle \nu, (\alpha'_{n,j})_\nu, a_j \rangle \in \mathcal{O}(n, j+1)$ but only if $j < m$. We also stipulate that $\mathcal{O}(n,j) \subseteq \mathcal{O}(n,j+1)$. We use $T'(n)$, α'_n and g_n to denote $T'(n,m)$, $\alpha'_{n,m}$ and $g_{n,m}$ respectively.

We now prove the counterparts to Lemmas 4.11 and 4.12, but we will need an extension of the former lemma. We will first prove a technical lemma which will help us. Sections of the proof are almost identical to those presented in Section 4.

Lemma 5.14. *Fix $n < \omega$. Then:*

(i) For all j such that $0 < j < m$ and $\langle \nu, (\alpha'_{n,j-1})_\nu, a \rangle \in \mathcal{O}(n,j) - \mathcal{O}(n,j-1)$, $a = a_{j-1}$.

(ii) For all $i < j \leq m$, $T'(n,i) \subseteq T'(n,j)$, and for all $\nu \in T'(n,i)$, $g_{n,i}(\nu) = g_{n,j}(\nu)$ and $\alpha'_{n,i}(\nu) = \alpha'_{n,j}(\nu)$.

(iii) For all $i < j \leq m$ and $\sigma \in G(n,j)$, there is a $\tau \in G(n,i)$ such that $\tau \subseteq \sigma$.

(iv) For all $j \leq m$ and $\kappa \in T(n, \mathcal{O}(n,j))$, if $\alpha_{n,\mathcal{O}(n,j)}(\kappa) = a_k$ then $k \leq j$; and if $\kappa \in T(n, \mathcal{O}(n,j)) - T(n, \mathcal{O}(n,j-1))$, then $\alpha_{n,\mathcal{O}(n,j)}(\kappa) = a_j$.

(v) For all $j \leq m$ and $\nu \in G(n,j)$, $\alpha_{n,\mathcal{O}(n,j)}(\nu) = a_j$.

(vi) $\mathcal{O}(n, m+1) - \mathcal{O}(n,m) = \emptyset$.

Proof. We prove (i)–(v) simultaneously by induction on j and then on $|\nu|$.

(i): If $j = 0$, there is nothing to show. Suppose that $j > 0$. The elements of $\mathcal{O}(n,j) - \mathcal{O}(n,j-1)$ are of the form $\langle \nu, (\alpha'_{n,j-1})_\nu, a_{j-1} \rangle$.

(ii),(iii): If $j = 0$ then there is nothing to show. Assume that $j > 0$, and fix $i < j$; by induction, we may assume without loss of generality that $i = j-1$. By (i) and as $\mathcal{O}(n,j-1) \subseteq \mathcal{O}(n,j)$, the definitions of $T'(n,j)$, $g_{n,j}$ and $\alpha'_{n,j}$ will agree with those for $T'(n,i)$, $g_{n,i}$ and $\alpha'_{n,i}$ until we reach an element $\nu \in T(n, \mathcal{O}(n,j-1))$ such that $\langle \nu, (\alpha'_{n,j-1})_\nu, a_{j-1} \rangle \in \mathcal{O}(n,j) - \mathcal{O}(n,j-1)$. Now neither ν nor any of its extensions are on $T'(n,j-1)$, nor will such a node be in the domain of $g_{n,j-1}$, so (ii) will hold. Furthermore, by the definition of $\mathcal{O}(n,j)$, $\nu \in G(n,j-1)$. Now given $\sigma \in G(n,j)$ the induction, as it proceeds for $\rho \subseteq \sigma$, will reach such a ν before it specifies that $\sigma \in G(n,j)$. So $\nu \subseteq \sigma$ and (iii) will hold.

(iv): As $\mathcal{O}(n,0) = \emptyset$, and as, by the definition of hybrid n-trees, the conclusion of (iv) must hold if for all κ such that $\langle \kappa, (\alpha'_n)_\kappa, a \rangle \in \mathcal{O}(n,j)$ it is the case that $a = a_k$ for some $k \leq j-1$, the first conclusion of (iv) follows from (i). Now if $\nu \in T(n, \mathcal{O}(n,j)) \cap T(n, \mathcal{O}(n,j-1))$ then we noted in the preceding paragraph that $\alpha_{n,\mathcal{O}(n,j)}(\nu) = \alpha_{n,\mathcal{O}(n,j-1)}(\nu)$, so by (ii), the conditions for placing $\nu \in T'(n,j-1)$ will be the same as those for placing $\nu \in T'(n,j)$. Thus if $\kappa \in T(n, \mathcal{O}(n,j)) - T(n, \mathcal{O}(n,j-1))$ and μ is the shortest substring of κ such that $\mu \in T(n, \mathcal{O}(n,j)) - T(n, \mathcal{O}(n,j-1))$, then $\mu \in G(n,j-1)$, so by (v) inductively, $\alpha_{n,\mathcal{O}(n,j)}(\mu) = a_j$. The second conclusion of (iv) now follows from the first conclusion and the definition providing the construction of hybrid n-trees.

(v): Suppose that $\nu \in G(n,j)$. By (iii) inductively, there is a $\sigma \subseteq \nu$ such that $\sigma \in G(n,j-1)$, so by Definition 5.13, we will place $\langle \sigma, (\alpha'_n)_\sigma, a_{j-1} \rangle \in \mathcal{O}(n,j)$. But then in $T(n, \mathcal{O}(n,j))$, we must have $\alpha_{n,\mathcal{O}(n,j)}(\nu) = a_k$ for some $k \geq j$. By (iv), $k \leq j$.

(vi): No triple $\langle \nu, (\alpha'_n)_\nu, a_m \rangle$ is ever placed in $\mathcal{O}(n,j)$ for any $j \leq m+1$. $\square$

Lemma 5.15. *Fix $n < \omega$. Then:*
(i) $G(n,j) \neq \emptyset$ for all $j \leq m$.
(ii) If $\eta \in G(n,j)$, then for all $k < \omega$, $\Phi_k(\oplus \widetilde{I}_{\eta,n,\alpha'_n}) \neq D_{\eta,n,(\alpha'_n)_\eta,(g_n)_\eta}$.

Proof. (i): Assume that (i) fails, and fix the smallest $j \leq m$ such that $G(n,j) = \emptyset$. Then $g = g_{n,j}$ labels all of $T'(n,j) = T(n, \mathcal{O}(n,j))$ which is a hybrid n-tree, so by the definition of $G(n,j)$, there is no copy of the module $M(n, \mathcal{O}(n,j), g)$ on T which has a (terminal) node τ satisfying either

$$\tau = \tau^{-\frown}\langle o \rangle \subset \Lambda \text{ for some } o \in \{w, r\},$$

or

$$\tau = \tau^{-\frown}\langle z \rangle \text{ and } \tau \text{ is uninitialized at cofinitely many stages.}$$

Now by our construction, there is a copy M of $M(n, \mathcal{O}(n,j), g)$ with initial node $\delta \subset \Lambda$. Let $\tau = \tau^{-\frown}\langle o \rangle \subset \Lambda$ be a terminal node of this module. Then $o \in \{w, r, z\}$, and by the counterpart of Lemma 4.4(ii), τ is uninitialized at cofinitely many stages, yielding the desired contradiction.

(ii): Fix $j \leq m$, $\eta \in G(n,j)$ and $k < \omega$. Then by the definition of $G(n,j)$, there is a terminal node ρ of a copy of some module $M(n, \widetilde{\mathcal{O}}, g)$ on T such that $\gamma(\rho^-) = \eta$, $g_{n,\widetilde{\mathcal{O}},g}(\gamma(\rho^-)) = k$, $(g_{n,\widetilde{\mathcal{O}},g)})_{\gamma(\rho^-)} = (g_n)_\eta$, $(\alpha_{n,\widetilde{\mathcal{O}},g)})_{\gamma(\rho^-)} = (\alpha'_n)_\eta$, and $\alpha_{n,\widetilde{\mathcal{O}},g}(\gamma(\rho^-)) = a_j$, and such that either $\rho \subset \Lambda$ and $\rho = \rho^{-\frown}\langle o \rangle$ for some $o \in \{w, r\}$, or ρ is uninitialized at cofinitely many stages, and $\rho = \rho^{-\frown}\langle z \rangle$. In the former case, (ii) follows from the counterpart of Lemma 4.8(ii),(iii), and in the latter case, (ii) follows from the counterpart of Lemma 4.9. $\square$

The next lemma is the counterpart of Lemma 4.12. While the proof of this lemma is similar to that of Lemma 4.12, we will present the proof, as biorderability is crucial to its success but played no role in the earlier proof.

Lemma 5.16. *Fix $n < \omega$ and $j \leq m$. Suppose that $\nu \in G(n,j)$. Then for all computable partial functionals Φ, $\Phi(\oplus \widetilde{I}_{\nu,n,\alpha'_n}) \neq D_{\nu,n,(\alpha'_n)_\nu,(g_n)_\nu}$ and for all $V \in \widetilde{F}_{\nu,n,\alpha'_n}$, $D_{\nu,n,(\alpha'_n)_\nu,(g_n)_\nu} \leq_T V$.*

Proof. Let $F = \widetilde{F}_{\nu,n,\alpha'_n}$, $I = \widetilde{I}_{\nu,n,\alpha'_n}$ and $D = D_{\nu,n,(\alpha'_n)_\nu,(g_n)_\nu}$. By the definition of $G_{n,j}$ and Lemma 5.15(ii), for all computable partial functionals Φ, $\Phi(\oplus I) \neq D$.

Suppose that $\rho \subset \nu$ and $\alpha'_n(\rho) = a_k$. As $\rho \subset \nu \in G(n,j)$, we must have $\rho \in T'(n,j) \subseteq T(n, \mathcal{O}(n,j))$, so by Lemma 5.14(ii),(iv) and the definition of $T'(n,j)$, $\alpha'_n(\rho) = \alpha_{n,\mathcal{O}(n,k)}(\rho)$ and $k \leq j$; furthermore, by the definition of $T'(n,k)$, $\rho \notin G(n,k)$. Hence if M is a copy, on T, of a module $M(n, \mathcal{O}, h)$ and $\tau \in T$ is a node of the implementation phase of M such that τ is uninitialized at cofinitely many stages, $\gamma(\tau^-) = \rho$, $h^+_{\gamma(\tau^-)} = (g_n)^+_\rho$, and $(\alpha_{n,\mathcal{O},h})^+_{\gamma(\tau^-)} = (\alpha'_n)^+_\rho$, then $\tau = \tau^{-\frown}\langle k \rangle$ for some $k \leq n$.

It remains to show that $D \leq_T V$ whenever $V \in F$. Let $\mathcal{M}$ be the set of all modules for the construction for requirements of the form $R_{n,\mathcal{O},h}$ where $\nu \in T(n, \mathcal{O}, h)$, $h_\nu = (g_n)_\nu$, and $(\alpha_{n,\mathcal{O},h})_\nu = (\alpha'_n)_\nu$.

Any number p which might enter D is either appointed as a follower for a module $M \in \mathcal{M}$ and is assigned to the tree of strategies before stage p, or cannot enter D. Suppose that p is such a follower. If $p \in D^p$, then $p \in D$. So suppose that $p \notin D^p$. We consider the two possible cases.

First assume that $V = C_b$ for some $b \in L$ such that $b \not\leq a_j$. As $a_j \in L_{MI}$, a permission witness q for b is appointed when p is appointed, and will enter its target set if and only if p enters its target set (and both will enter at the same stage). Hence $p \in D$ iff $q \in V$.

Now suppose that $V = W_m \in F$. (M is a subset of T, so we may take α as the corresponding selection map.) Fix the unique nodes η of the implementation phase of M and κ of the design phase of M such that $\gamma(\eta) = \gamma(\kappa) = \nu$, and the unique node β of M at which the implementation phase begins, noting that $\kappa \subset \beta \subseteq \eta$. As $V = W_m \in F$, the definition of the function δ implies that there is a node ξ such that $\beta \subseteq \xi \subset \xi^\frown\langle m\rangle \subseteq \eta$ and $\alpha(\xi) \geq \alpha(\eta)$; hence by (5.1), $\alpha(\xi) \leq_d \alpha(\eta)$ and $\alpha(\xi) \leq_p \alpha(\eta)$. Thus there is a node σ of M such that $\sigma \subset \kappa$ and $\gamma(\xi) = \gamma(\sigma)$. Fix k such that $\gamma(\sigma) = \alpha_k$.

Fix the smallest stage $s \geq p$ such that $W_m^s \restriction p+1 = W_m \restriction p+1$. We assume that $p \notin D^s$ and p is still a follower, else we are done. We now claim that there is a smallest stage $t \geq s$ at which one of the following occurs:

(i) $p \in D^t$.
(ii) κ is initialized at stage t.
(iii) β is initialized at stage t.
(iv) For some ρ such that $\beta \subseteq \rho \subset \eta$, $\lambda_t \supseteq \rho^\frown\langle o\rangle$ for some $o \neq z$ such that $\rho^\frown\langle o\rangle \nsubseteq \eta$.

To see the claim, assume that (ii)–(iv) fail for all $t \geq s$. As $\sigma \subset \kappa$ and (ii) fails for all $t \geq s$, u_k^s, p_k^s and σ_k^s are defined, and for all $t \geq s$, $u_k^t = u_k^s$, $p_k^t = p_k^s$ and $\sigma_k^t = \sigma_k^s$. Now the failure of (iii) implies that there is a longest ρ such that $\beta \subseteq \rho \subseteq \eta$ which is uninitialized at some stage $s_0 \geq s$. We showed in the second paragraph of the proof that it cannot be the case that $\rho \subset \eta$ and $\rho^\frown\langle z\rangle \subseteq \lambda_t$ for infinitely many $t \geq s_0$. Hence by Remark 4.1, there will be a $t \geq s$ and an outcome $o \neq z$ such that $\rho^\frown\langle o\rangle \subseteq \lambda_t$. Now the failure of (iv) and the maximality of ρ imply that $\rho = \eta$. As $p \notin D^s$, η must have been initialized at stage s, so t is the first stage $\geq s$ at which η is uninitialized. We now see that p cannot have been released before stage t, so (3.5) cannot hold. Thus (3.4) must hold and we place $p \in D^t$, and so (i) holds.

We now search for the first $t \geq s$ at which one of the clauses (i)–(iv) holds. Note that t can be effectively determined from W_m. If (i) holds, then $p \in D$. If (ii) holds, then p is canceled and never again reappointed as a follower, so $p \notin D$. If (iii) holds but (ii) fails to hold, then by the counterpart to Lemma 4.7 and as $\sigma \subset \kappa$ implies $u_j^s < p$, $p \notin D$. Finally, if (iv) holds but (ii) and (iii) fail to hold, then by the counterpart to Lemma 4.2(ii), $p \notin D$. We have thus shown that $D \leq_T V = W_m$. □

Lemma 5.16 allows us to conclude only that, for a node $\nu \in G(n,j)$, the requirement assigned to ν is satisfied if we set $\alpha(\nu) = a_j$. The next lemma will show that once we reach the terminal node of $\widetilde{T}(n)$, we will be able to piece together coherent sets of ideals for which all requirements are satisfied for all $a \in L_{MI}$.

Lemma 5.17. *Fix $n < \omega$. Suppose that $\nu \in G(n,m)$. Then for all $j \leq m$, there is a $\eta_j \subseteq \nu$ such that $\eta_j \in G(n,j)$ and for all partial functionals Φ, $\Phi(\oplus\widetilde{I}_{\eta_j,n,\alpha'_n}) \neq D_{\eta_j,n,(\alpha'_n)_{\eta_j},(g_n)_{\eta_j}}$ and for all $V \in \widetilde{F}_{\eta_j,n,\alpha'_n}$, $D_{\eta_j,n,(\alpha'_n)_{\eta_j},(g_n)_{\eta_j}} \leq_T V$.*

Proof. Fix $j \leq m$. By Lemma 5.14(iii), there is a $\eta_j \subseteq \nu$ such that $\eta_j \in G(n,j)$. The conclusion of the lemma now follows from Lemma 5.16. □

Definition 5.18. For each $\nu \in G(n,m)$ and each $j \leq m$, let $\chi_{\nu,n,j}$ be the characteristic function of the subset $\widehat{\delta}(\nu, \alpha'_n \restriction \nu, a_j)$ of $\{0,\ldots,n\}$. Define the 2^{m+1}-branching tree X of sequences of $(m+1)$-tuples of elements of $\{0,1\}$ by

$$\langle \delta_0,\ldots,\delta_m\rangle \in X \leftrightarrow \exists n \exists \nu \in G(n,m) \forall j \leq m (\delta_j = \chi_{\nu,n,j}).$$

Note that the elements of X are $(m+1)$-tuples of strings of the same length. By Lemma 5.15(i), X is an infinite tree. Hence by König's Lemma, X has an infinite branch $\Gamma = \langle \Gamma^0,\ldots,\Gamma^m\rangle$, (each of whose components is the characteristic function of a subset of ω). For all $n < \omega$ and $j \leq m$, let $\Gamma^j_n = \Gamma^j \restriction n+1$.

For $a \in L$, define

$$\widehat{I}_a = \{C_b : b \in L\} \cup \{W_i : i < \omega\} \text{ if } a = 1,$$

$$\widehat{I}_a = \{C_b : b \leq a\} \cup \{W_v : \Gamma^j(v) = 0\}$$

if $a = a_j \in L_{MI}$, and

$$\widehat{I}_a = \bigcap \{\widehat{I}_b : b \geq a \;\&\; b \in L_{MI}\}$$

otherwise. We again note that in the above definitions, the sets $\widehat{I}_a$ have, as elements, the names of computably enumerable sets rather than the sets themselves. (Thus the same computably enumerable set may appear infinitely often with a different name each time; the name will either have the form W_i or C_b.) For each $a \in L$, let I_a be the set of degrees of elements of $\widehat{I}_a$.

We will show that the sets $\widehat{I}_a$ are degree-invariant and the map f described in (2.7) induces a degree-preserving pseudolattice homomorphism. We will prove a useful lemma first.

Lemma 5.19. *Fix $j \leq m$ and $k < \omega$. Then there are $n \geq k$ and $\sigma \in G(n,j)$ such that $\widehat{I}_{a_j} \cap \{W_0,\ldots,W_k\} = \widetilde{I}_{\sigma,n,\alpha'_n} \cap \{W_0,\ldots,W_k\}$. Furthermore, σ has the property that there is a c.e. set D for which $D \not\leq_T \oplus \widetilde{I}_{\sigma,n,\alpha'_n}$, $D \leq_T W_v$ for all $v \leq n$ such that $W_v \notin \widetilde{I}_{\sigma,n,\alpha'_n}$, and $D \leq_T C_b$ if $C_b \notin \widetilde{I}_{\sigma,n,\alpha'_n}$.*

Proof. By the definition of X and as $|\Gamma| = \infty$, there must be $n \geq k$ and $\nu \in G(n,m)$ such that

$$W_i \in \widehat{I}_{a_j} \Leftrightarrow \Gamma^j_n(i) = 0 \Leftrightarrow \chi_{\nu,n,j}(i) = 0 \Leftrightarrow i \notin \widehat{\delta}(\nu, \alpha'_n \restriction \nu, a_j)$$

for all $i \leq k$. By Lemma 5.17, we can fix $\sigma \in G(n,j)$ and a c.e. set D such that $\sigma \subseteq \nu$, $D \not\leq_T \oplus \widetilde{I}_{\sigma,n,\alpha'_n}$, $D \leq_T W_v$ for every $v \leq n$ such that $W_v \notin \widetilde{I}_{\sigma,n,\alpha'_n}$ and $D \leq_T C_b$ if $C_b \notin \widetilde{I}_{\sigma,n,\alpha'}$. Now

$$W_i \in \widetilde{I}_{\sigma,n,\alpha'_n} \Leftrightarrow i \notin \widehat{\delta}(\sigma, \alpha'_n \restriction \sigma, a_j).$$

By Lemma 5.14(iv),(v), we can apply Remark 5.5 to conclude that

$$i \notin \widehat{\delta}(\sigma, \alpha'_n \restriction \sigma, a_j) \Leftrightarrow i \notin \widehat{\delta}(\nu, \alpha'_n \restriction \nu, a_j).$$

The lemma now follows. □

Lemma 5.20. *The above definition induces a homomorphism of $\mathcal{R}_p$ onto $\mathcal{L}$.*

Proof. We first show that the filters $\widehat{I}_a$ are degree invariant. By the way these filters are generated, it suffices to consider the case in which $a \in L_{MI}$. Fix j such that $a = a_j$. We show the degree invariance for $W_k \equiv_T W_i$; the proof for $C_b \equiv_T W_i$ follows analogously, replacing W_k with C_b and eliminating k. Fix k and i such that $W_k \equiv_T W_i$, and assume that $k > i$. We next fix σ and D as provided by Lemma 5.19. As $W_k \equiv_T W_i$, it follows that

$$D \leq_T W_i \Leftrightarrow D \leq_T W_k,$$

and so by the properties of D,

$$W_i \in \widetilde{I}_{\sigma,n,\alpha'_n} \Leftrightarrow W_k \in \widetilde{I}_{\sigma,n,\alpha'_n}.$$

Thus by Lemma 5.19,

$$W_i \in \widehat{I}_a \Leftrightarrow W_k \in \widehat{I}_a.$$

(2.1) and (2.4) follow easily from the definition of $\widehat{I}_a$ and the degree-invariance of these sets; and (2.5) follows easily from Lemma 5.19, Remark 3.4 and the degree-invariance of the sets $\widehat{I}_a$.

We now verify (2.2). Fix $b, c \in L_{MI}$ such that $b < c$. (Note that (2.2) requires us to consider only $b, c \in L_{MI}$.) By Remark 5.5, it follows that if $\Gamma_c(m) = 0$ then $\Gamma_b(m) = 0$, and so by the placement of the sets C_a into the various sets $\widetilde{I}$, that $\widehat{I}_b \subseteq \widehat{I}_c$ and hence that $I_b \subseteq I_c$.

It remains to verify (2.3). Fix $a, b \in L$ such that $a \not\geq b$. Then $C_b \in \widehat{I}_b - \widehat{I}_a$, whence (2.3) follows from the degree-invariance of these sets. □

We have thus proved:

Theorem 5.21. *Let $\mathcal{L}$ be a finite distributive lattice such that L_{MI} is biorderable. Then there is a pseudolattice homomorphism from $\mathcal{R}_p$ onto $\mathcal{L}$.*

We now describe some finite distributive lattices for which L_{MI} is biorderable, and some which do not have such biorderings. We note that any finite distributive lattice $\mathcal{L}$ is uniquely determined by its associated set L_{MI}, and for any finite poset P, there is a finite lattice $\mathcal{L}$ whose associated poset L_{MI} (under the ordering of $\mathcal{L}$) is isomorphic to $\mathcal{P}$. Hence it suffices to describe lattices by specifying the associated poset L_{MI}.

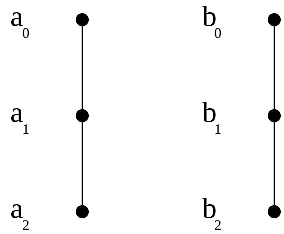
Corollary 5.22. *Let $\mathcal{L}$ be a linearly ordered set with universe $a_0 > a_1 > \cdots > a_m$. Then there is a pseudolattice homomorphism from $\mathcal{R}_p$ onto $\mathcal{L}$.*

Proof. It is easily checked that the orderings $a_0 <_d a_1 <_d \cdots <_d a_m$ and $a_0 <_p a_1 <_p \cdots <_p a_m$ satisfy (5.1) and (5.2). The result now follows from Theorem 5.21. □

The above orderings can also be used for the case in which we can partition L_{MI} into a set of chains such that elements from different chains are incomparable. (Figure 5.1 pictures a poset L_{MI} with this property.) We call such a lattice *linearly decomposable*. We just require that the order of stacking for $<_p$ reverses the order for $<_d$ for elements of different chains, and use the ordering of Corollary 5.22 within each maximal chain.

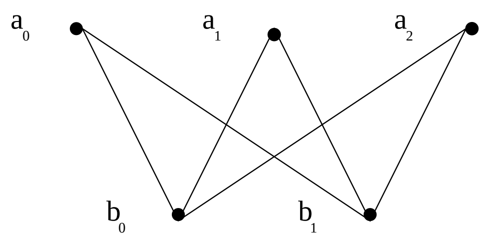
Hence:

Corollary 5.23. *Let $\mathcal{L}$ be a linearly decomposable finite distributive lattice. Then there is a pseudolattice homomorphism from $\mathcal{R}_p$ onto $\mathcal{L}$.*



Linearly decomposable
Figure 5.1

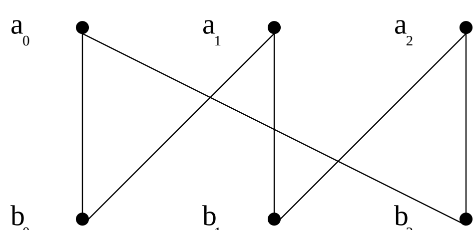
We say that L is *incomparably decomposable* if the elements of L_{MI} can be partitioned into a collection of antichains $A_0, \dots, A_k$ such that whenever $i < j$, $a \in A_i$ and $b \in A_j$, it is the case that $a > b$. (Figure 5.2 pictures a poset L_{MI} with this property.) We fix arbitrary orderings on each A_i, use this ordering for $<_d$ and its reverse for $<_p$, and stack these orderings by $A_i <_d A_j$ if $i < j$ and $A_i <_p A_j$ if $i < j$. The resulting orderings satisfy (5.1) and (5.2). Hence:



Incomparably decomposable
Figure 5.2

Corollary 5.24. *Let $\mathcal{L}$ be an incomparably decomposable finite distributive lattice. Then there is a pseudolattice homomorphism from $\mathcal{R}_p$ onto $\mathcal{L}$.*

Definition 5.25. We say that L_{MI} has an *n-crown* if L_{MI} can be partitioned into two sets $A = \{a_i : i < n\}$ and $B = \{b_i : i < n\}$ which satisfy exactly the following ordering relations: $b_i < a_i, a_{i+1}$ if $i \in \{0, \dots, n-2\}$ and $b_{n-1} < a_0, a_{n-1}$.



3-crown
Figure 5.3

L_{MI} cannot be biorderable if it has an n-crown for some $n \geq 3$. For suppose otherwise. Then by symmetry, we may assume that $a_0 <_d a_j$ for all $j > 0$; in particular, $a_0 <_d a_1$. Suppose that $a_0 <_d a_j <_d a_1$ for some $j > 1$ in order to obtain a contradiction. By (5.2), $a_1 <_p a_j <_p a_0$. Now $b_0 \leq a_0, a_1$, so by (5.1), $a_0 <_d b_0$ and $a_1 <_p b_0$. As $b_0|a_j$, it follows from (5.2) that either $b_0 <_d a_j$ or $b_0 <_p a_j$. But in the first case we would have $a_j <_d a_1 <_d b_0 <_d a_j$, and in the second case we would have $a_j <_p a_0 <_p b_0 <_p a_j$, both of which yield contradictions. An inductive argument of this type now shows that $a_0 <_d a_1 <_d \cdots <_d a_{n-1}$. By (5.2), $a_{n-1} <_p \cdots <_p a_1 <_p a_0$. As $b_{n-1}|a_1$, it follows from (5.2) that

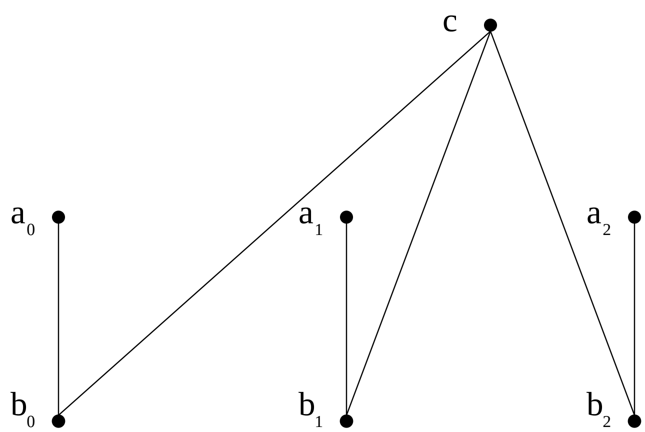
$$b_{n-1} <_d a_1 \leftrightarrow a_1 <_p b_{n-1}.$$

Now by (5.1), $a_1 <_d a_{n-1} <_d b_{n-1}$ and $a_1 <_p a_0 <_p b_{n-1}$, yielding a contradiction. We have just shown that:

Proposition 5.26. *If $\mathcal{L}$ is a finite distributive lattice such that L_{MI} has an n-crown for some $n \geq 3$, then L_{MI} is not biorderable.*

We present one more example of a non-biorderable poset.

Definition 5.27. A poset P has an *upwards n-check* if there are $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-1}$ and c in P satisfying exactly the following order relations: $b_i < a_i, c$ for all $i < n-1$. P has a *downwards n-check* if the poset with universe P obtained by interchanging $<$ and $>$ has an upwards n-check. P has an *n-check* if P has either an upwards or downwards n-check. If P has an n-check for some $n > 0$ with elements as above, then we call c the *central element* of the n-check.



Upwards 3-check
Figure 5.4

L_{MI} cannot be biorderable if it has an n-check for some $n \geq 3$, and hence a 3-check. For suppose otherwise. By symmetry, we may assume that the 3-check is upwards. Note that any permutation of $\{0, 1, 2\}$ produces an isomorphic poset, a property which we will refer to below as *symmetry*. By (5.1), $a_i, c <_d b_i$ and $a_i, c <_p b_i$ for all $i \leq 2$. By symmetry, we may assume that $a_0 <_d a_1 <_d a_2$, whence by (5.2), $a_2 <_p a_1 <_p a_0$. As $b_i|a_j$ whenever $i \neq j$, it will follow from the above inequalities and (5.2) that

$$a_0 <_d b_0 <_d a_1 <_d b_1 <_d a_2 <_d b_2$$

and

$$a_2 <_p b_2 <_p a_1 <_p b_1 <_p a_0 <_p b_0.$$

By (5.1), $c <_d b_0 <_d a_1$ and $c <_p b_2 <_p a_1$; but this contradicts (5.2) as $c|a_1$. We have just shown that:

Proposition 5.28. *If $\mathcal{L}$ is a finite distributive lattice such that L_{MI} has a 3-check, then L_{MI} is not biorderable.*

We have identified two more minimal examples of posets that are not biorderable. The first starts with the two crown with elements a_0, a_1, b_0, b_1 satisfying $a_0|a_1$, $b_0|b_1$ and $b_i < a_j$ for all $i, j \leq 1$, and adds two new elements c_0 and c_1, both incomparable with a_1, such that $b_i < c_i < a_0$ for $i \leq 1$. The second modifies the three check of Figure 5.4 by reversing the check determined by a_0 and b_0, i.e., specifying instead that $a_0, c < b_0$ and preserving all other comparability relationships not contradicted by this condition. We conjecture that any minimal biordering (under embeddability) is either one of our examples or the dual of one of these examples.

Our ultimate goal is to characterize the finite lattices which are isomorphic to quotients of $\mathcal{R}_p$. It seems plausible to us, for a finite distributive lattice $\mathcal{L}$ with universe L, that L_{MI} is biorderable iff $\mathcal{L}$ is isomorphic to a quotient of $\mathcal{R}_p$. In the non-distributive case, additional requirements, governing intersections of ideals, must be satisfied. We do not know if there are quotients of $\mathcal{R}_p$ which are isomorphic to any finite non-distributive lattices.

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