Embedding Finite Lattices into the Ideals of Computably Enumerable Turing Degrees

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Abstract

We show that the lattice L_{20} is not embeddable into the lattice of ideals of computably enumerable Turing degrees ($\mathcal{I}$). We define a structure called a pseudolattice that generalizes the notion of a lattice, and show that there is a Π_2 necessary and sufficient condition for embedding a finite pseudolattice into $\mathcal{I}$.

1 The Lattice Embedding Problem

The problem of determining which finite lattices are embeddable into the upper semilattice of computably enumerable Turing degrees ($\mathcal{R}$) has been studied intensively for over thirty years. A solution to the embedding problem would be an important step toward a decision procedure for the $\forall\exists$-theory of $\mathcal{R}$.

Calhoun began the investigation of embeddings into the lattice of ideals of computably enumerable Turing degrees ($\mathcal{I}$) in [C90], where he gave a sufficient condition for embedding a finite lattice into $\mathcal{I}$. Clearly, any lattice that is embeddable into $\mathcal{R}$ is embeddable into $\mathcal{I}$. However, the embedding problem for $\mathcal{I}$ is not the same as the embedding problem for $\mathcal{R}$. While Lachlan and Soare [LS80] showed that the lattice S_8 is not embeddable into

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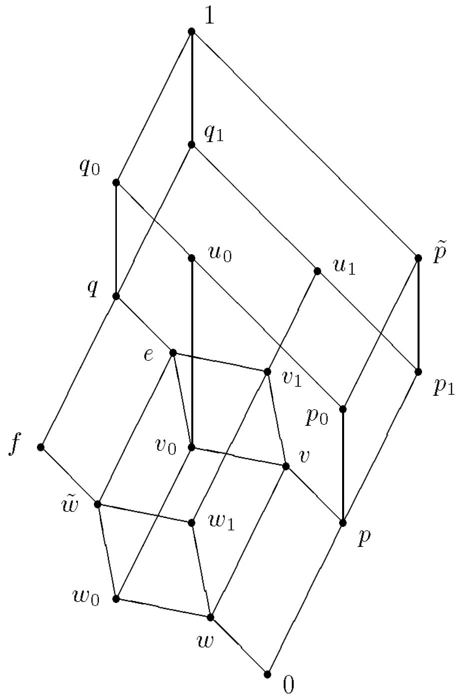

Figure 1: Lattice L_{20}

$\mathcal{R}$, Calhoun showed that S_8 is embeddable into $\mathcal{I}$. In fact, S_8 is embeddable into any nontrivial initial segment of $\mathcal{I}$ as was shown by Weber [Wb94]. Weber also gave a sufficient condition for embedding a finite lattice into any nontrivial initial segment of $\mathcal{I}$.

Until the present work, no finite lattice was known *not* to be embeddable into $\mathcal{I}$. For $\mathcal{R}$, Ambos-Spies and Lerman [AL86] generalized Lachlan and Soare's proof for S_8 to obtain a non-embeddability condition, NEC. Until 1997 all finite lattices that were known not to be embeddable into $\mathcal{R}$ satisfied NEC. Lattices that satisfy NEC contain a *critical triple*: pairwise incomparable elements a, b, and c such that $a \vee b = a \vee c$ and $b \wedge c \leq a$. (Downey introduced the term *critical triple* in [D90]. Weinstein independently gave an equivalent definition in [Wn88].) For lattices that satisfy NEC, the combination of the critical triple with an infimum results in an obstruction to constructing an embedding into $\mathcal{R}$. Since this type of obstruction can be overcome in constructing an embedding into $\mathcal{I}$, some computability theorists (including the authors) were led to think that every finite lattice might be embeddable into $\mathcal{I}$. However, we show that this is not the case.

Lempp and Lerman [LL97] constructed a finite lattice, L_{20}, without critical triple that cannot be embedded into $\mathcal{R}$. We show that L_{20} is also not embeddable into $\mathcal{I}$. The proof is based on the proof in [LL97]. However, we organize the proof in a different way to show that any embedding of five specified elements of L_{20} into $\mathcal{R}$ can be extended to an upper semilattice embedding that is incompatible with an embedding of L_{20} into $\mathcal{R}$. This version of the proof may also be used to show that L_{20} is not embeddable into $\mathcal{I}$. The construction is simpler in some ways than the one in [LL97] since we do not need to consider infimum requirements directly.

In the last section we introduce a structure called a *pseudolattice* that generalizes the notion of a lattice. We show that there is a Π_2 necessary and sufficient condition for embedding a finite pseudolattice into $\mathcal{I}$. The proof uses the techniques developed for the proof that L_{20} is not embeddable into $\mathcal{I}$ to reduce the pseudolattice embedding problem for $\mathcal{I}$ to an embedding question about $\mathcal{R}$. We then use results of Lerman [L98] to obtain a Π_2 condition.

2 A Property of $\mathcal{R}$ That Implies L_{20} is Not Embeddable Into $\mathcal{I}$

2.1 The property S

We will use a property S of the c.e. degrees to show that L_{20} is not embeddable into $\mathcal{I}$. Property S captures the essentials of the argument used in [LL97] to show that L_{20} is not embeddable into $\mathcal{R}$. The property has been stated carefully so that an embedding of L_{20} into $\mathcal{I}$ would imply that S is false in $\mathcal{R}$.

Definition 2.1 *Property S: For all f, p_0, p_1, w_0, w_1 where $p_0 \vee w_1 \geq f$ and $p_1 \vee w_0 \geq f$ there exist b_0, b_1, c_0, c_1 such that the following inequalities hold:*

$$b_0 \leq w_0 \vee w_1 \vee p_0 \tag{2.1}$$
$$b_0 \leq p_0 \vee p_1 \tag{2.2}$$
$$b_1 \leq w_0 \vee w_1 \vee p_1 \tag{2.3}$$
$$b_1 \leq p_0 \vee p_1 \tag{2.4}$$
$$c_0 \leq w_0 \vee w_1 \vee p_1 \tag{2.5}$$
$$c_0 \leq w_0 \vee b_0 \tag{2.6}$$
$$c_1 \leq w_0 \vee w_1 \vee p_0 \tag{2.7}$$
$$c_1 \leq w_1 \vee b_1 \tag{2.8}$$

$$\textit{and } c_0 \vee c_1 \quad \geq \quad f \tag{2.9}$$

To see why S can be used to obtain non-embeddability results for L_{20} one may consider trying to find elements b_0, b_1, c_0 and c_1 in L_{20} satisfying equations (2.1)-(2.9). The five elements f, p_0, p_1, w_0 and w_1 of L_{20} are the most important ones in the proof in [LL97]. The five elements can also be joined to obtain other elements of L_{20} as follows: $w_0 \vee w_1 \vee p_0 = q_0$, $p_0 \vee p_1 = \tilde{p}$ and $w_0 \vee w_1 \vee p_1 = q_1$. These joins appear on the right hand sides of inequalities (2.1)-(2.4) of property S. If inequalities (2.1)-(2.4) of property S are interpreted in L_{20} they imply that $b_0 \leq q_0 \wedge \tilde{p} = p_0$ and $b_1 \leq q_1 \wedge \tilde{p} = p_1$. Then if inequalities (2.5)-(2.8) are interpreted in L_{20} they imply that $c_0 \leq q_1 \wedge u_0 = v_0$ and $c_1 \leq q_0 \wedge u_1 = v_1$. Finally, inequality (2.9) implies that $v_0 \vee v_1 \geq f$, but this last inequality is false in L_{20}.

2.2 It is Sufficient to Show that S Holds in $\mathcal{R}$

We now show that if there were an embedding of L_{20} into $\mathcal{I}$ then S would be false in $\mathcal{R}$. The argument formalizes the intuition given in the previous paragraph.

Lemma 2.2 *If L_{20} is embeddable in $\mathcal{I}$ then S is false in $\mathcal{R}$.*

Proof: Suppose there is an embedding of L_{20} into $\mathcal{I}$. For each a in L_{20}, let $\mathbf{I}_a$ be the image of a under the embedding. Since $f \not\leq e$, there is a degree $\mathbf{d}_f \in \mathbf{I}_f \setminus \mathbf{I}_e$. Since $\mathbf{I}_{p_0} \vee \mathbf{I}_{w_1} \geq \mathbf{I}_f$, there are degrees $\mathbf{d}_{p_0} \in \mathbf{I}_{p_0}$ and $\mathbf{d}_{w_1} \in \mathbf{I}_{w_1}$ such that $\mathbf{d}_{p_0} \vee \mathbf{d}_{w_1} \geq \mathbf{d}_f$. Similarly, there are degrees $\mathbf{d}_{p_1} \in \mathbf{I}_{p_1}$ and $\mathbf{d}_{w_0} \in \mathbf{I}_{w_0}$ such that $\mathbf{d}_{p_1} \vee \mathbf{d}_{w_0} \geq \mathbf{d}_f$.

Now suppose that S is true in $\mathcal{R}$. Then there is a degree $\mathbf{d}_{b_0}$ corresponding to b_0, where $\mathbf{d}_{b_0} \leq \mathbf{d}_{w_0} \vee \mathbf{d}_{w_1} \vee \mathbf{d}_{p_0}$ and $\mathbf{d}_{b_0} \leq \mathbf{d}_{p_0} \vee \mathbf{d}_{p_1}$. It follows that

$$\mathbf{d}_{b_0} \in (\mathbf{I}_{w_0} \vee \mathbf{I}_{w_1} \vee \mathbf{I}_{p_0}) \cap (\mathbf{I}_{p_0} \vee \mathbf{I}_{p_1}) = \mathbf{I}_{q_0} \cap \mathbf{I}_{\tilde{p}} = \mathbf{I}_{p_0}.$$

Still assuming that S is true in $\mathcal{R}$, there must be a degree $\mathbf{d}_{c_0}$ corresponding to c_0 where $\mathbf{d}_{c_0} \leq \mathbf{d}_{w_0} \vee \mathbf{d}_{w_1} \vee \mathbf{d}_{p_1}$ and $\mathbf{d}_{c_0} \leq \mathbf{d}_{w_0} \vee \mathbf{d}_{b_0}$. It follows that

$$\mathbf{d}_{c_0} \in (\mathbf{I}_{w_0} \vee \mathbf{I}_{w_1} \vee \mathbf{I}_{p_1}) \cap (\mathbf{I}_{w_0} \vee \mathbf{I}_{p_0}) = \mathbf{I}_{q_1} \cap \mathbf{I}_{u_0} = \mathbf{I}_{v_0}$$

For S to be true there must also be degrees $\mathbf{d}_{b_1}$ and $\mathbf{d}_{c_1}$ corresponding to b_1 and c_1. By symmetry, $\mathbf{d}_{c_1} \in \mathbf{I}_{v_1}$, so $\mathbf{d}_{c_0} \vee \mathbf{d}_{c_1} \in \mathbf{I}_{v_0} \vee \mathbf{I}_{v_1} = \mathbf{I}_e$. However, since $\mathbf{d}_f \notin \mathbf{I}_e$, we see that $\mathbf{d}_{c_0} \vee \mathbf{d}_{c_1} \not\geq \mathbf{d}_f$. But then $\mathbf{d}_{c_0}$ and $\mathbf{d}_{c_1}$ do not satisfy (2.9) when substituted for c_0 and c_1.

3 Proof of S in $\mathcal{R}$

3.1 The Construction

In this section we will show that S holds in $\mathcal{R}$. The proof is similar to the proof in [LL97] that L_{20} is not embeddable into $\mathcal{R}$. However, since S does not explicitly use the infimum operation, we will not need to build infima. The proof involves building sets corresponding to variables b_0, b_1, c_0 and c_1 in S and reductions to ensure inequalities (2.1)-(2.9).

Lemma 3.1 *Property S is true in $\mathcal{R}$.*

Proof: To show that S holds, let F, P_0, P_1, W_0, W_1 be any c.e. sets, where

$$P_0 \vee W_1 \geq F \text{ via } \Omega_0 \text{ and } P_1 \vee W_0 \geq F \text{ via } \Omega_1.$$

We will build c.e. sets B_0, C_0, B_1, C_1 and reductions Δ_0, Λ_0, Δ_1, Λ_1, Γ_0, Ξ_0, Γ_1, Ξ_1, Φ such that

$$\begin{aligned}
B_0 &\leq W_0 \vee W_1 \vee P_0 \text{ via } \Delta_0 && (3.1)\\
B_0 &\leq P_0 \vee P_1 \text{ via } \Lambda_0 && (3.2)\\
B_1 &\leq W_0 \vee W_1 \vee P_1 \text{ via } \Delta_1 && (3.3)\\
B_1 &\leq P_0 \vee P_1 \text{ via } \Lambda_1 && (3.4)\\
C_0 &\leq W_0 \vee W_1 \vee P_1 \text{ via } \Gamma_0 && (3.5)\\
C_0 &\leq W_0 \vee B_0 \text{ via } \Xi_0 && (3.6)\\
C_1 &\leq W_0 \vee W_1 \vee P_0 \text{ via } \Gamma_1 && (3.7)\\
C_1 &\leq W_1 \vee B_1 \text{ via } \Xi_1 && (3.8)\\
\text{and } C_0 \vee C_1 &\geq F \text{ via } \Phi && (3.9)
\end{aligned}$$

The idea is to show that we can resolve the conflict between building Φ and the other reductions. For each number x targeted for F we will appoint traces y_0 targeted for C_0 and y_1 targeted for C_1. We will need to show that if x ever enters F we can reflect the change by putting y_0 in C_0 or y_1 in C_1.

3.2 Notation

If we perform a calculation at stage s we will intend that all the reductions and sets involved refer to the approximation at stage s. If upper case Greek letter Φ denotes a reduction then lower case Greek letter ϕ denotes the use of Φ. The use in computing $\Phi^{A \vee B}(x)$ is written $\phi(x, A, B)$ and is a disjoint union of two sets: the use of A and the use of B. If we only want the part of the use that refers to A we will write $\phi(x, A)$. We will choose uses in a monotonic fashion so that if $y < x$ then $\phi(y, A) \subseteq \phi(x, A)$. We also extend the definition of use to finite sets of natural numbers by taking the union of the uses of each member of the set. Thus if X is a finite set of natural numbers, $\phi(X, A) = \bigcup_{x \in X} \phi(x, A)$.

3.3 The Reduction Condition

At a stage s when the Φ-traces y_0 and y_1 of x are appointed we will ensure the following condition:

$$\begin{aligned}
\omega_0(x, P_0, W_1) &\subseteq \gamma_1(y_1, P_0, W_1) && (3.10)\\
\omega_0(x, P_0, W_1) &\subseteq \delta_0(\xi_0(y_0, B_0), P_0, W_1) && (3.11)\\
\omega_1(x, P_1, W_0) &\subseteq \gamma_0(y_0, P_1, W_0) && (3.12)\\
\omega_1(x, P_1, W_0) &\subseteq \delta_1(\xi_1(y_1, B_1), P_1, W_0) && (3.13)
\end{aligned}$$

and either (I) or (II) holds.

(I)

$$\begin{aligned}
\omega_1(x, W_0) &\subseteq \xi_0(y_0, W_0) && (3.14)\\
\omega_1(x, P_1) &\subseteq \lambda_0(\xi_0(y_0, B_0), P_1) && (3.15)\\
\omega_1(x, P_1) &\subseteq \lambda_1(\xi_1(y_1, B_1), P_1) && (3.16)
\end{aligned}$$

(II)

$$\begin{aligned}
\omega_0(x, W_1) &\subseteq \xi_1(y_1, W_1) && (3.17)\\
\omega_0(x, P_0) &\subseteq \lambda_1(\xi_1(y_1, B_1), P_0) && (3.18)\\
\omega_0(x, P_0) &\subseteq \lambda_0(\xi_0(y_0, B_0), P_0) && (3.19)
\end{aligned}$$

Lemma 3.2 *Let s be any stage at which the reduction condition holds. Then if x enters F at stage s we can (eventually) enumerate y_0 in C_0 or y_1 in C_1.*

Proof: If x enters F then a number in $\omega_0(x, P_0, W_1)$ must enter P_0 or W_1 and a number in $\omega_1(x, P_1, W_0)$ must enter P_1 or W_0. There are eight cases depending on which sets the numbers enter and which of (I) or (II) holds in the reduction condition. For cases 1-4 suppose (I) holds.

Cases 1 and 2: If a number in $\omega_1(x, P_1)$ enters P_1 and a number in $\omega_0(x, P_0, W_1)$ enters P_0 or W_1 we proceed as follows. By (3.11)

$$\Delta_0^{W_0 \vee W_1 \vee P_0}(\xi_0(y_0, B_0))\uparrow$$

and by (3.15)

$$\Lambda_0^{P_0 \vee P_1}(\xi_0(y_0, B_0))\uparrow.$$

We enumerate a number in $\xi_0(y_0, B_0)$ into B_0 making $\Xi_0^{W_0 \vee B_0}(y_0)\uparrow$. Also $\Gamma_0^{W_0 \vee W_1 \vee P_1}(y_0)\uparrow$ by (3.12). So we can enumerate y_0 into C_0.

Cases 3 and 4: If a number in $\omega_0(x, W_1)$ enters W_1 and a number in $\omega_0(x, P_0, W_1)$ enters P_0 or W_1 we proceed as follows. By (3.12)

$$\Gamma_0^{W_0 \vee W_1 \vee P_1}(y_0)\uparrow$$

and by (3.14)

$$\Xi_0^{W_0 \vee B_0}(y_0)\uparrow.$$

We enumerate y_0 into C_0.

Cases 5,6,7 and 8: The four cases where (II) holds are symmetric to the cases where (I) holds.

3.4 Maintaining the Reduction Condition

We must see that the reduction condition can be maintained.

Lemma 3.3 *We can ensure that the reduction condition holds at any stage at which $\omega_0(x, P_0, W_1)$ and $\omega_1(x, P_1, W_0)$ are both defined.*

Proof: We must show that if numbers in $\omega_0(x, P_0, W_1)$ enter P_0 or W_1 or if numbers in $\omega_1(x, P_1, W_0)$ enter P_1 or W_0, then we can recover the reduction condition. Note that the previous lemma shows that if the uses of $\Omega_0^{P_0 \vee W_1}$ and $\Omega_1^{P_1 \vee W_0}$ both change, then we could enumerate y_0 into C_0 or y_1 into C_1. We could then appoint new traces satisfying the reduction condition.

We must therefore consider the eight cases in which either (I) or (II) holds in the reduction condition and numbers in a relevant use enter one of the sets P_0, W_1, P_1, W_0. For cases 1-4 suppose (I) holds.

Cases 1 and 2: If one or more numbers in $\omega_0(x, P_0, W_1)$ enter P_0 or W_1 then by (3.10)

$$\Gamma_1^{W_0 \vee W_1 \vee P_0}(y_1)\uparrow$$

and by (3.11)

$$\Delta_0^{W_0 \vee W_1 \vee P_0}(\xi_0(y_0, B_0))\uparrow.$$

To recover the reduction condition, we wait for $\Omega_0^{P_0 \vee W_1}(x)$ to converge and then choose new uses for Γ_1 and Δ_0 large enough to ensure (3.10) and (3.11). The reduction condition will then hold with (I).

Case 3: If one or more numbers in $\omega_1(x, P_1)$ enter P_1 but no number in $\omega_1(x, W_0)$ enters W_0 then by (3.12)

$$\Gamma_0^{W_0 \vee W_1 \vee P_1}(y_0)\uparrow,$$

by (3.13)

$$\Delta_1^{W_0 \vee W_1 \vee P_1}(\xi_1(y_1, B_1))\uparrow,$$

by (3.15)

$$\Lambda_0^{P_0 \vee P_1}(\xi_0(y_0, B_0))\uparrow$$

and by (3.16)

$$\Lambda_1^{P_0 \vee P_1}(\xi_1(y_1, B_1))\uparrow.$$

To maintain the reduction condition we must ensure that (3.10)-(3.13) hold and that either (I) or (II) holds. In this case, we will no longer be able to ensure that (I) holds, but we can now ensure that (II) holds. We enumerate a number in $\xi_1(y_1, B_1)$ into B_1 to make $\Xi_1^{W_1 \vee B_1}(y_1)\uparrow$. To recover the reduction condition, we wait for $\Omega_1^{P_1 \vee W_0}(x)$ to converge and choose uses of Γ_0 and Δ_1 large enough to ensure (3.12) and (3.13). Since there are no relevant changes to W_1 or P_0, (3.10) and (3.11) will hold and the left hand sides of the inequalities in (II) will not change. Thus we can also choose uses of Ξ_1, Λ_0, and Λ_1 large enough to ensure that (II) holds. Therefore the reduction condition will hold with (II).

Case 4: If one or more numbers in $\omega_1(x, W_0)$ enter W_0 then by (3.12)

$$\Gamma_0^{W_0 \vee W_1 \vee P_1}(y_0)\uparrow$$

and by (3.14)

$$\Xi_0^{W_0 \vee B_0}(y_0)\uparrow.$$

We enumerate y_0 into C_0. To recover the reduction condition we wait for $\Omega_1^{P_1 \vee W_0}(x)$ to converge and then appoint new traces, ${y_0}^*$ and ${y_1}^*$ and define uses large enough to satisfy the reduction condition. (In this case we can make both (I) and (II) hold.)

Cases 5,6,7 and 8: The four cases where (II) holds are symmetric to the four cases where (I) holds.

3.5 Showing Our Reductions are Total

To complete the proof, we must show that the reductions we build are total.

Lemma 3.4 *Assuming that $P_0 \vee W_1 \geq F$ via Ω_0 and $P_1 \vee W_0 \geq F$ via Ω_1, we can ensure that Δ_0, Λ_0, Δ_1, Λ_1, Γ_0, Ξ_0, Γ_1, Ξ_1 and Φ are all total.*

Proof: First consider the reduction condition corresponding to one number x targeted for F. The only situations in which we must enumerate and redefine the traces y_0 or y_1 are in response to either x entering F or changes in the uses of Ω_0 or Ω_1 in computing x. If x enters F we will no longer need traces for x. If x never enters F, then since Ω_0 and Ω_1 are total, there will be only finitely many changes in the uses of Ω_0 and Ω_1 in computing x, so eventually we will not have to change the traces y_0 and y_1. Thus we can ensure that Φ is total in this situation. Similarly, the reduction condition will only cause us to increase the size of uses of Δ_0, Λ_0, Δ_1, Λ_1, Γ_0, Ξ_0, Γ_1 and Ξ_1 when the use of Ω_0 or Ω_1 in computing x changes. Thus we can ensure that all these reductions are total in this situation.

Finally, consider the full situation in which there are infinitely many reduction conditions corresponding to the infinitely many numbers targeted for F. For a given natural number x we need to see that the uses of our reductions in computing x converge. Since we choose uses monotonically we can assume by induction that there is a stage at which the uses of our reductions in computing any $y < x$ have settled down to their final values. After that stage we will need to change the uses of our reductions in computing x only finitely many times as discussed in the previous paragraph. Thus all of our reductions are total. This completes the proof that S holds in $\mathcal{R}$.

4 A Π_2 Necessary and Sufficient Condition for Embedding a Finite Pseudolattice into $\mathcal{I}$

In this section, we will show that there is a Π_2 necessary and sufficient condition for embedding a finite pseudolattice into $\mathcal{I}$. The study of embeddings into $\mathcal{L}$ reduces to listing a certain set of requirements, and determining whether those requirements can be satisfied. In particular, when proving nonembeddability results, one specifies a finite set of requirements which cannot be satisfied. Normally, one does not use all the lattice relationships to get the set of requirements which obstruct embeddability; one focuses on minimal set of relationships such that any structure possessing elements with these relationships is nonembeddable. Collections of such relationships are best expressed in the language of pseudolattices.

The underlying idea of our proof is to reduce the problem to an embedding question about $\mathcal{R}$. We begin with a more careful analysis of embedding results for $\mathcal{R}$, an analysis which applies to effectively generated sets of pseudolattice conditions (the lattice conditions are then a special case) and looks at homomorphisms as a generalization of embeddings. We then show that such a set of conditions is consistent with the existence of a homomorphism into $\mathcal{I}$ if a second set of conditions effectively generated from the first and yielding a ranked system, is consistent with the existence of a homomorphism into $\mathcal{R}$. This will enable us to reduce the question of the existence of a homomorphism (and hence an embedding) into $\mathcal{I}$ to one of the existence of a homomorphism into $\mathcal{R}$, where we have a Π_2 condition by Lerman [L98].

Definition 4.1 *We will be looking at algebraic structures on which relations $M_n(u_1, \ldots, u_n, b)$ and $J_n(u_1, \ldots, u_n, b)$ for $n \geq 2$ are defined, and which have an ordering, $\leq$, of their universe. For notational convenience, we will represent the sequence $\langle u_1, \ldots, u_n \rangle$ as U^n, and will thus write the above relations as $M_n(U^n, b)$ and $J_n(U^n, b)$ respectively. We say that the sequence $V^k = \langle v_1, \ldots, v_k \rangle$* strongly covers *$U^n$ if for every $i \leq n$ there is a $j \leq k$ such that $v_j \leq u_i$. We say that V^k* covers *U^n if for every $i \leq n$, either there is a $j \leq k$ such that $v_j \leq u_i$, or there is a subset $Y^m = \langle y_1, \ldots, y_m \rangle$ of V^k (i.e., a set of coordinates of V^k which we allow to be reordered) such that $J_k(Y^k, u_i)$.*

We now define the structures which we call pseudolattices. These are relational approximations to lattices which reflect the types of requirements which must be satisfied for embeddings into $\mathcal{R}$. We begin with a partial ordering relation $\leq$ which we axiomatize with the first three conditions. As we will be looking at homomorphisms rather than embeddings, and want to preserve some (but not necessarily all) of the $\not\leq$ relationships, we introduce another relation, $\not\preceq$ which will be a subset of $\not\leq$, as specified in the fourth condition below. M_n will be a *pseudomeet* relation, i.e., $M_n(a_1, \ldots, a_n, b)$ will reflect the statement saying that every element $\leq$ all the a_i is $\leq b$; and similarly, J_n will be a *pseudojoin* relation, i.e., $J_n(a_1, \ldots, a_n, b)$ will reflect the statement saying that every element $\geq$ all the a_i is $\geq b$. These properties are axiomatized by the fifth and seventh conditions. If a pseudomeet relationship is specified for a sequence, then we want a similar statement specified for a second sequence which is strongly covered by the first; and if a pseudojoin relationship is specified for a sequence, then we want a similar statement specified for a second sequence which covers the first. These properties are axiomatized by the sixth and eighth conditions. Finally, if there is a pseudojoin condition for a sequence, we want our structure to have an element which represents the true join of the sequence, as specified by the last condition.

Definition 4.2 *A* pseudolattice *is a structure* $\mathcal{L} = \langle L, \leq, \not\leq, \{J_n : n \geq 2\}, \{M_n : n \geq 2\}\rangle$ *satisfying the following conditions:*

$$a \leq a \tag{4.1}$$

$$a \leq b \;\&\; b \leq a \;\rightarrow\; a = b \tag{4.2}$$

$$a \leq b \;\&\; b \leq c \;\rightarrow\; a \leq c \tag{4.3}$$

$$a \not\leq b \;\rightarrow\; \neg(a \leq b) \tag{4.4}$$

$$M_n(a_1, \ldots, a_n, b) \;\rightarrow\; \forall c(c \leq a_1 \;\& \ldots \&\; c \leq a_n \;\rightarrow\; c \leq b) \tag{4.5}$$

$$M_n(U^n, b) \;\&\; U^n \text{ strongly covers } V^k \;\&\; b \leq d \;\rightarrow\; M_k(V^k, d) \tag{4.6}$$

$$J_n(a_1, \ldots, a_n, b) \;\rightarrow\; \forall c(a_1 \leq c \;\& \ldots \&\; a_n \leq c \;\rightarrow\; b \leq c) \tag{4.7}$$

$$J_n(U^n, b) \;\&\; V^k \text{ covers } U^n \;\&\; d \leq b \;\rightarrow\; M_n(V^k, d) \tag{4.8}$$

$$J_n(a_1, \ldots, a_n, b) \;\rightarrow\; \exists d(a_1 \leq d \;\& \ldots \&\; a_n \leq d \;\&\; J_n(a_1, \ldots, a_n, d)) \tag{4.9}$$

A close examination of the portion of a typical embedding construction for $\mathcal{R}$ corresponding to a finite set of requirements, leads to the observation that the requirements specify a finite pseudolattice $\mathcal{P}$ under the natural closure conditions (4.6) and (4.8), and that this portion of the construction, in isolation, shows that there is a pseudolattice homomorphism from $\mathcal{P}$ into $\mathcal{R}$. Homomorphisms are defined in the standard algebraic way, as specified below.

Definition 4.3 *A* homomorphism *from a pseudolattice* $\mathcal{P}$ *with universe* P *into a pseudolattice* $\mathcal{Q}$ *with universe* Q *is a function* $f : P \rightarrow Q$ *which satisfies the following conditions:*

$$a \leq b \;\rightarrow\; f(a) \leq f(b) \tag{4.10}$$

$$a \not\leq b \;\rightarrow\; f(a) \not\leq f(b) \tag{4.11}$$

$$M_n(a_1, \ldots, a_n, b) \;\rightarrow\; M_n(f(a_1), \ldots, f(a_n), f(b)) \tag{4.12}$$

$$J_n(a_1, \ldots, a_n, b) \;\rightarrow\; J_n(f(a_1), \ldots, f(a_n), f(b)) \tag{4.13}$$

If $\not\leq$ *and* $\not\leq$ *are identical on* P*, then* f *is an* embedding. f *is a* lattice embedding *if, in addition,* $\mathcal{P}$ *is order-isomorphic to a lattice, and the following conditions hold:*

$$\text{If } a \vee b = c, \text{ then the relation } J_2(a,b,c) \text{ holds in } \mathcal{P} \tag{4.14}$$

$$\text{If } a \wedge b = c, \text{ then the relation } M_2(a,b,c) \text{ holds in } \mathcal{P} \tag{4.15}$$

A principally decomposable finite pseudolattice is one for which the tracing function in a pinball machine construction can be required to be finite at any stage. (For lattices, this is equivalent to having no critical triple.)

Definition 4.4 *A pseudolattice* $\mathcal{P}$ *with universe* P *is* principally decomposable *if whenever* $b < a$*, and the interval* $(b,a) = \emptyset$*,* V^k *is a subset of* $\{c : c \leq a\} - \{c : c \leq b\}$*, and there is a* d *such that* $M_k(V^k, d)$*, then there is a* $c \in \{c : c \leq a\} - \{c : c \leq b\}$ *such that* $c \leq d$*.*

A ranked pseudolattice is a principally decomposable pseudolattice in which the number of hereditary traces (in a pinball machine construction) for an element of the universe is uniformly bounded, independent of stage. We define the notion of rank.

Definition 4.5 *Let* $\mathcal{P}$ *be a principally decomposable pseudolattice on universe* P *and let* c *be a join-irreducible element of* A *(i.e., if* $a_1, \ldots, a_n \leq c$ *and* $J_n(a_1, \ldots, a_n, c)$*, then* $c = a_i$ *for some* i*).* c *has* rank 1 *if whenever* $J_n(a_1, \ldots, a_n, c)$ *with all* a_i *join-irreducible then* $c \leq a_i$ *for some* i*. By induction,* c *has* rank n+1 *if* c *does not have rank* $\leq n$ *and whenever* $J_n(a_1, \ldots, a_n, c)$ *with all* a_i *join-irreducible, then each* a_i *has rank* $\leq n$*.* $\mathcal{P}$ *is* ranked *if every join-irreducible element of* P *has a rank.*

Lerman [L00] gives an example of a lattice L_{11} which is principally decomposable but is not ranked. In [L98], Lerman finds a Π_2 necessary and sufficient condition, GBTA, for the embeddability of a finite ranked lattice into $\mathcal{R}$, and in [L00], he extends this condition to finite principally decomposable lattices. A careful examination of those proofs leads to the observation that the fact that the structures being embedded are lattices is not important; rather, one deals with effectively generated sequences of requirements for incomparability, comparability, pseudojoin, and pseudomeet relations. (In fact, we can require that the sequence be $\mathbf{0}'$-presentable rather than the stronger condition of effective generation.) Furthermore, the embeddings constructed are pseudolattice homomorphisms, and the analysis is really a pseudolattice homomorphism analysis. The conditions for a GBTA only deal with M_2, as no other pseudomeet relations are needed in the lattice setting. We can modify the conditions of GBTA to handle $M_n(a_1, \dots, a_n, b)$ in the obvious way, i.e., whenever $M_2(p, q, r)$ is mentioned and the condition of GBTA requires some property of both p and q, then the revised condition requires the same property of all a_i; and if the condition of GBTA requires some property of either p or q, then the revised condition requires the same property of some a_i. These are the only modifications required in the proof, so we note, from the proof of [L98], that:

Theorem 4.6 *There is a pseudolattice homomorphism from an effectively generated ranked pseudolattice $\mathcal{P}$ into $\mathcal{R}$ iff $\mathcal{P}$ satisfies GBTA as revised above.*

While we will only need to appeal to Theorem 4.6 in this paper, we note that the proof of [L00] generalizes in the same way:

Theorem 4.7 *There is a pseudolattice homomorphism from an effectively generated principally decomposable pseudolattice $\mathcal{P}$ into $\mathcal{R}$ iff $\mathcal{P}$ satisfies GBTA as revised above.*

We now consider the techniques for constructing homomorphisms from a finite pseudolattice $\mathcal{P}$ into $\mathcal{I}$. The homomorphism will map each $a \in P$ (P is the universe of $\mathcal{P}$) into an ideal I_a of $\mathcal{I}$. I_a is specified by building an increasing sequence $\langle A_a^i : i \in \mathcal{N} \rangle$ of computably enumerable sets such that $\mathbf{b} \in I_a$ iff there is a set B of degree $\mathbf{b}$ such that $B \leq_T A_a^n$ for some $n \in \mathcal{N}$.

Comparability Requirements: If $a \leq b$, then we require that $I_a \subseteq I_b$. We achieve this by satisfying the requirements

$$A_a^i \leq_T A_b^i$$

for all $i \in \mathcal{N}$.

Incomparability Requirements: If $a \not\leq b$, then we require that $I_a \not\subseteq I_b$. We achieve this by fixing a number $j_0(a, b)$, and satisfying the requirements

$$A_a^{j_0(a,b)} \neq \Phi_e(A_b^i)$$

for all $e, i \in \mathcal{N}$.

Join Requirements: Suppose that $U^n = \{u_1, \dots, u_n\} \subseteq P$ is a set of join-irreducible elements such that $J_n(U^n, c)$ and no $e \in U^n$ satisfies $c \leq e$. By (4.9), fix d such that $u_1, \dots, u_n \leq d$ and $J_n(U^n, d)$. We then require that $I_c \subseteq I_d$. We achieve this by fixing a number $j_1(i, U^n)$ and defining a computable partial functional Δ_{c,U^n}^i to satisfy the requirements

$$A_c^i = \Delta_{c,U^n}^i(\oplus\{A_e^{j_1(i,U^n)} : e \in U^n\})$$

for all $i \in \mathcal{N}$.

Meet Requirements: Suppose that $V^n = \{v_1, \ldots, v_n\} \subseteq P$ and $M_n(V^n, b)$. We require that $I_{v_1} \cap \ldots \cap I_{v_n} \subseteq I_b$. By Posner's Lemma [EP78], it suffices to fix a number $j_2(e, i, V^n)$ and to construct computable partial functionals Δ^i_{e,V^n} to satisfy the requirements

$$\Phi_e(A^i_{v_1}) \; = \; \ldots \; = \Phi_e(A^i_{v_n}) \text{ total } \rightarrow \; \Phi_e(A^i_{v_1}) \; = \Delta^i_{e,V^n}(A_b^{j_2(e,i,U^n)})$$

for all $e, i \in \mathcal{N}$.

We effectively order all the requirements listed above as $\{R_i : \; i \in \mathcal{N}\}$. Once the ordering of requirements is given, we indicate how to choose j_i for the above requirements. We proceed by induction on the ordering of requirements, defining j_i if it is not already defined. For incomparability requirements, we choose $j_0(a, b) = 0$. For join requirements, we choose $j_1(i, U^n)$ to be the smallest integer which is greater than all indices mentioned in preceding requirements; and for meet requirements, we choose $j_2(e, i, U^n)$ to be the smallest integer which is greater than all indices mentioned in preceding requirements. Now closing under the pseudomeet and pseudojoin conditions and the comparability conditions specified by the individual ideals, we have a pseudolattice which is called the *augmented pseudolattice* for this sequence of requirements. We observe that the augmented pseudolattice is effectively generated from the sequence of requirements, and that if we start with a finite initial segment of this sequence of requirements, then we generate a finite subpseudolattice of the augmented pseudolattice, which we call the *augmented pseudolattice for the finite sequence of requirements.* We note that these finite pseudolattices are ranked. For if n is the largest number such that a set of the form A^n_a is placed into the augmented pseudolattice, then all elements A^n_a have rank 1, all elements A^{n-1}_a have rank ≤ 2, etc. We now present the necessary and sufficient condition for the existence of a homomorphism from $\mathcal{P}$ into $\mathcal{I}$.

Theorem 4.8 *There is a homomorphism from $\mathcal{P}$ into $\mathcal{I}$ iff the pinball machine games for each of the augmented subpseudolattices of $\mathcal{P}$ generated by a finite sequence of requirements containing only one incomparability requirement, that requirement being the last element of the sequence, has a good blocked target array.*

Proof: The reverse implication follows from Lerman [L98]. Conversely, suppose that there is a homomorphism f from $\mathcal{P}$ into $\mathcal{I}$. For each $p \in P$, let I_p be generated by the increasing sequence of computably enumerable sets $\mathcal{A}_p = \{A_p^n : n \in \mathcal{N}\}$. We note that any infinite subsequence of $\mathcal{A}_p$ also generates I_p. By the existence of f, it follows from Lerman [L98] that any subsequence of the sequence of requirements in which only the final requirement is an incomparability requirement has a good blocked target array (gbta), i.e., there is a choice of levels j_i for the requirements for which the gbta exists; we represent these choices as those given by $\tilde{j}_i$. We show that we can force $j_i = \tilde{j}_i$ by thinning out the increasing sequences of sets representing the ideals. The thinning-out process proceeds by induction on the indices of the requirements in the sequence, except that the single incomparability requirement is handled first. We begin by fixing the level j_0 for the incomparability requirement, and thinning out the ideals by removing all sets A_p^j for $j < j_0$. Thus without loss of generality, we may assume that $\tilde{j}_0(a,b) = 0$. Note that we always have $A_p^i \leq_T A_p^{i+1}$. Suppose, by induction, that we encounter a join requirement for which the embedding produces a level $\tilde{j}_1(i, U^n)$ while our algorithm produces $j_1(i, U^n)$. If $j_1(i, U^n)$ has been defined for an earlier requirement, then by induction, $j_1(i, U^n) = \tilde{j}_1(i, U^n)$. Otherwise, we compare $j_1(i, U^n)$ and $\tilde{j}_1(i, U^n)$. If $j_1(i, U^n) = \tilde{j}_1(i, U^n)$, we continue the induction, noting that the induction hypothesis is satisfied. If $\tilde{j}_1(i, U^n) < j_1(i, U^n)$, then as $A_p^i \leq_T A_p^{i+1}$ for all i and p, the embedding would also be witnessed if we had defined $\tilde{j}_1(i, U^n)$ to be $j_1(i, U^n)$, so again we continue the induction, noting that the induction hypothesis is satisfied. Finally, if $\tilde{j}_1(i, U^n) > j_1(i, U^n)$, then by thinning out the representations of the ideals by removing A_p^i for all $p \in P$ and $i \in [j_1(i, U^n), \tilde{j}_1(i, U^n))$, we again see that we have $\tilde{j}_1(i, U^n) = j_1(i, U^n)$ with respect to the new representation for the ideals, while preserving the similar equalities for the requirements treated earlier in the induction. The case in which we encounter a meet requirement is treated as in the preceding paragraph, replacing j_1 with j_2. We conclude that the conditions generated by the requirements are satisfiable by a homomorphism into $\mathcal{I}$. By closing these conditions under the required properties for $\leq$, pseudojoin and pseudomeet, we generate a ranked pseudolattice which can be embedded into $\mathcal{R}$. By Lerman [L98], all sequences of requirements for this pseudolattice as described in the condition of the theorem have gbtas.

We note that the homomorphism condition provided by Theorem 4.6 is a Π_2 condition.

Remark 4.9 *Clearly, an embedding into $\mathcal{R}$ is an embedding into $\mathcal{I}$. Lachlan and Soare [LS80] have shown that the lattice S_8 is not embeddable into $\mathcal{R}$, while Calhoun [C90] has shown that this lattice is embeddable into $\mathcal{I}$. Thus the necessary and sufficient conditions for embeddability into $\mathcal{I}$ and into $\mathcal{R}$*

do not coincide. It is not known whether these conditions coincide on smaller classes of lattices such as principally decomposable lattices or ranked lattices.

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