A FINITE LATTICE WITHOUT CRITICAL TRIPLE THAT CANNOT BE EMBEDDED INTO THE ENUMERABLE TURING DEGREES

Steffen Lempp
Manuel Lerman

Department of Mathematics
University of Wisconsin
Madison, WI 53706–1388, USA

Department of Mathematics
University of Connecticut
Storrs, CT, 06269–3009, USA

Abstract. We exhibit a finite lattice without critical triple that cannot be embedded into the enumerable Turing degrees. Our method promises to lead to a full characterization of the finite lattices embeddable into the enumerable Turing degrees.

0. Introduction. The search for a decision procedure for the $\forall\exists$-theory of the poset, $\mathbf{E}$, of (recursively) enumerable degrees is considered to be one of the major open problems of computability theory. Attempts at finding decision procedures have concentrated on deciding fragments of this theory, fragments which are generally existential theories of $\mathbf{E}$ in expanded languages. Most of the efforts have centered around a particular fragment, namely, the one obtained by adding a constant symbol 0 (representing least element), a binary relation symbol $\vee$ (representing join), and $(n+1)$-ary predicates $M_n(a_0, \dots, a_{n-1}, b)$ for all $n \geq 2$ which are defined by $\forall x(x \leq a_0 \;\&\ldots\&\; x \leq a_{n-1} \to x \leq b)$. (As the meet of enumerable degrees does not always exist, these predicates are meant to capture as much of the meet operation as is feasible.) A structure in this language is called a *partial lattice with least element*.

The study of (finite) partial lattice embeddings into $\mathbf{E}$ was begun by Lachlan [La1] and Yates [Y] almost thirty years ago. Since that time, many embedding and some non-embeddability results have been obtained, but no characterization of the finite partial lattices which can be embedded into $\mathbf{E}$ has been found. The most

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general results for the lattice setting were obtained by Ambos-Spies and Lerman [AL1,AL2]; they presented a sufficient condition, NEC, for non-embeddability, and a sufficient condition, EC, for embeddability, but were unable to show that the two conditions were complementary. (A discussion of the obstructions encountered in embedding proofs whose analysis led to these conditions can be found in [L2].) Subsequently, several people (including both authors) had conjectured that these conditions are, in fact, complementary. It is the purpose of this paper to show that this is not the case; we exhibit a non-embeddable twenty-element lattice, L_{20}, which fails to satisfy NEC. Our theorem also contradicts Downey's conjecture that a finite lattice is embeddable into every non-trivial interval of $\mathbf{E}$ iff it has no critical triples.

Two types of obstructions are encountered when trying to implement the pinball machine technology introduced in [L1] to embed lattices into $\mathbf{E}$. The first type of obstruction is captured by NEC, a condition formulated by Ambos-Spies and Lerman [AL1]. (Lachlan and Soare [LaS] had previously provided the first example, S_8, of a non-embeddable lattice, and their proof of its non-embeddability was the source of the intuition for the formulation of NEC.) This obstruction arises when the procedure for satisfying a join requirement requires the use of an infinitary trace procedure which endangers the satisfaction of a single meet requirement. The other type of obstruction arises from the interaction of a join requirement with several meet requirements, and necessitates retargeting traces for new sets. The new target causes potential injury to a meet requirement. All previous examples of lattices which gave rise to the second type of obstruction were lattices which also had an obstruction of the first type, and so satisfied NEC. Through a process of formulating stronger conditions than EC which were satisfied by all the embeddable lattices which we had hitherto examined and then trying to construct a finite lattice which failed to satisfy both this condition and NEC, we succeeded, after several iterations, in constructing such a lattice which was non-embeddable. Our method of proof can be generalized, and provides new insight towards obtaining a necessary and sufficient condition (in terms of the complementarity of two recursion theoretic constructions) for the embeddability of a finite partial lattice with least element into $\mathbf{E}$. (Note that every lattice has partial lattice structure).

Our notation generally follows that of Soare [S]. We abbreviate $X \upharpoonright (x+1)$ by $X[x]$. Upper case Greek letters will denote computable partial functionals, and the corresponding lower case letter denotes its use function. We assume, without loss of generality, that whenever we fix all but one argument x of a use function $\phi(\bar{a}, x)$, then the resulting function of one variable will be non-decreasing. We say that there is an *injury* to $\Psi(A;x)$ at stage s if $\Psi^{s-1}(A^{s-1};x)$ is defined with use $\psi(x, s-1)$, and $A^{s-1}[\psi(x,s-1)] \neq A^s[\psi(x,s-1)]$. If $A = B \oplus C$, then we will want to specify the set whose change causes the injury; thus for $D \in \{B, C\}$, we say that there is a *D-injury* to $\Psi(A;x)$ at stage s if $\Psi^{s-1}(A^{s-1};x)$ is defined with use $\psi(x,s-1)$, and $D^{s-1}[\psi(x,s-1)] \neq D^s[\psi(x,s-1)]$. (Here, the use is computed separately for each component of the direct sum.) These definitions will be used when the functional Ψ and the set A are given. If Ψ is being constructed and A is given, then the axioms being defined at stage $s-1$ will generally have the form $\Psi^s(A^{s-1};x)$. We make the obvious modification to the definition of *injury* in this case.

1. NEC and L_{20}. The conditions EC and NEC, mentioned in the introduction, are not central to this paper. As we deal only with a single non-embeddable lattice, the complicated condition, EC, is not needed. And we do not need the full power of NEC; we merely use the fact that every lattice which satisfies NEC has a critical triple, and will note that the lattice we present has no critical triples.

The isolation of critical triples from NEC was done independently by Downey [D] and Weinstein [W] in the pursuit of finding a necessary and sufficient condition ensuring the embeddability of a finite lattice into all intervals of **E**.

Definition. A triple $\langle a, b, c\rangle$ of elements of a finite lattice L is called a *critical triple* if a, b, and c are pairwise-incomparable, $a \vee b = a \vee c$ and $b \wedge c \leq a$.

The existence of critical triples in a lattice was shown, by Ambos-Spies and Lerman [AL1], to be equivalent to another property which is easier to verify in most situations; it is this latter variant of the definition which we will use in this paper. For convenience, we state the condition of [AL1] and prove that it is equivalent to the non-existence of critical triples in a finite lattice.

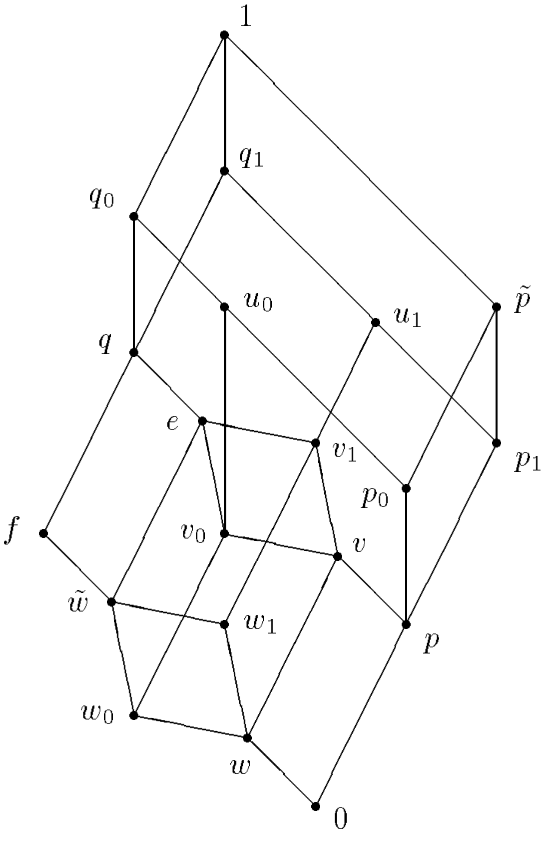

Figure 1: Lattice L_{20}

Proposition. *A finite lattice L with least element 0 fails to have critical triples if for all $a < d$ in L such that the interval (a, d) is empty, the difference of the intervals $[0, d] - [0, a]$ has a (unique) least element.*

Proof. First suppose that L has a critical triple $\langle a, b, c\rangle$. Let $d = a \vee b = a \vee c$; by replacing a, if necessary, by a maximal element in $[a, d)$, we can assume without loss of generality that d is a minimal cover of a. Now $[0,d]-[0,a]$ cannot have a unique least element e, else then $e \leq b, c$ so $e \leq b \wedge c$, and $e \not\leq a$, contrary to the definition of critical triple.

Conversely, suppose that there are $a < d \in L$ such that d is a minimal cover of a and $[0,d]-[0,a]$ does not have a least element. Then there are minimal elements $b|c \in [0,d]-[0,a]$. As $b, c \not\leq a$ and d is a minimal cover of a, $a \vee b = a \vee c = d$; and by the minimality of b and c, $b \wedge c \leq a$. Hence $\langle a, b, c\rangle$ is a critical triple. □

We now verify that the lattice L_{20} of Figure 1 has no critical triples.

Theorem 1.1. *L_{20} is a lattice which fails to have critical triples.*

Proof. We apply the proposition. The table below lists all possible choices for d and a such that d is a minimal cover of a, and the least element b of $[0,d]-[0,a]$.

d	a	b	d	a	b	d	a	b
1	q_0	p_1	1	q_1	p_0	1	$\tilde{p}$	w
q_0	q	p_0	q_0	u_0	w_1	$\tilde{p}$	p_1	p_0
q_1	q	p_1	q_1	u_1	w_0	$\tilde{p}$	p_0	p_1
q	e	f	q	f	p	u_0	p_0	w
u_0	v_0	p_0	u_1	v_1	p_1	u_1	p_1	w
p_1	p	p_1	p_0	p	p_0	e	v_0	w_1
e	v_1	w_0	e	$\tilde{w}$	p	v_0	v	w_0
v_0	w_0	p	v_1	v	w_1	v_1	w_1	p
f	$\tilde{w}$	f	v	w	p	v	p	w
p	0	p	$\tilde{w}$	w_0	w_1	$\tilde{w}$	w_1	w_0
w_1	w	w_1	w_0	w	w_0	w	0	w

□

In the remainder of the paper, we will show that L_{20} cannot be embedded into $\mathbf{E}$. We now present the type of analysis which leads to the proof. There are only six join and meet facts about L_{20} which are used in the proof, so the proof extends to any partial lattice which is order-isomorphic to L_{20} and satisfies these facts. They are: $p_0 \vee w_1 \geq f$; $p_1 \vee w_0 \geq f$; $q_0 \wedge u_1 \leq v_1$; $q_1 \wedge u_0 \leq v_0$; $q_0 \wedge \tilde{p} \leq p_0$; and $q_1 \wedge \tilde{p} \leq p_1$.

In order to embed a lattice L into $\mathbf{E}$, we construct an enumerable set A_c for each $c \in L$, and satisfy requirements which ensure that the correspondence yields a lattice isomorphism. The various isomorphism-preserving requirements impose certain restrictions on the construction. Suppose that $b, c, d \in L$. If $b \leq c$, then we require that there be a computable set C such that $A_b = A_c \cap C$. If $b \not\leq c$, then there will be stages of our construction when we will be placing numbers into A_b while permanently restraining other numbers from A_c. If $b \vee c = d$, then whenever a number x is a candidate to be placed into A_b, we will appoint a trace y for this number which must enter A_c or A_d at least as soon as x enters A_b; and if y enters its target set earlier than x, then a replacement trace for y must immediately be appointed.

If $b \wedge c = d$, then we must effectively compute a function g from A_d as it is separately effectively computed from A_b and A_c. The computation process will be revised as new numbers separately enter A_b and A_c. When a number enters, say,

A_b, we must not let A_c change its computation of g until a new computation from A_b is recovered, unless A_d is also allowed to revise its computation, else this strategy will fail. When a number enters A_b, it may raise the use of the computation of g from A_b on some number x; after a new computation is found, a number may enter A_c and thereby raise the use of the computation of $g(x)$ from A_c. We have now formed a *dangerous interval*; numbers entering A_d which lie below both new uses and above the use for the computation from A_d will now allow the computations of $g(x)$ from both A_b and A_c to change simultaneously, thereby allowing the value of $g(x)$ to change, without the ability to correct the computation from A_d. Thus we must prevent numbers in dangerous intervals from entering A_d.

We now see that there are potential conflicts between the strategies to satisfy join or diagonalization requirements, and the strategy to satisfy meet requirements. Meet requirements impose a restriction which prevents numbers captured in dangerous intervals from entering their target sets; if traces appointed for join requirements which are hereditarily related to low priority diagonalization requirements must be captured in these intervals, they cannot enter their respective target sets. This may ultimately force us to abandon all attempts to satisfy a fixed diagonalization requirement. The above intuition gave rise to the speculation by Lerman that the lattice S_8 might not be embeddable. Lachlan and Soare [LS] then showed that these conflicts were fatal as the lattice is nonembeddable. Their proof focused on a single meet requirement which could be forced, through diagonalization, to form dangerous intervals. A diagonalization requirement which required infinitely many traces because of associated join requirements could be forced to appoint traces in dangerous intervals, and thus can never be satisfied.

There is another point in the standard pinball model of an embedding construction where we could potentially force traces to be appointed *late* and so be unable to avoid dangerous intervals; this point is when several traces move to a new gate and must have their entry into their target sets separated by an expansionary stage for the gate. This can force the appointment of new traces with new targets (we call this procedure *retargeting*), some of which must enter their target sets at stages later than traces which had previously been appointed but have not yet entered their target sets. All previous examples where this occurred were lattices which satisfied NEC, so they witnessed the problems arising in the previos paragraph. L_{20} is an example which shows that such problems can arise even when NEC fails.

These various restrictions fatally conflict when we try to embed L_{20} into $\mathbf{E}$. Let us first motivate L_{20} and show that in some sense it is "the smallest example" of this phenomenon. We informally define a finite lattice to be a "length-n" lattice if any number x targeted for a set A corresponding to some join-irreducible element a of the lattice needs at most $n-1$ traces at any point of a construction. (Formally, we require that any minimal prime filter containing a can be generated (under upward closure) by a sequence $\overline{b}$ of length at most $n-1$ such that any initial segment of $\overline{b}$ also generates a prime filter in L.) It is now easy to see that the length-1 lattices are exactly the distributive lattices, which are known to be embeddable independently by Lerman and Thomason [T]. It is not too hard to see that all length-2 lattices are embeddable. So the "smallest" nonembeddable lattice must be at least length-3. L_{20} is not only length-3 but has an even stronger property: The two traces p_0 and p_1 required for f are interchangeable. An intermediate version of L_{20} (which was the starting point for our search leading to L_{20}) is the lattice obtained from L_{20} by deleting the points between w and u_0 and between w and u_1. The remaining points were later added to create a lot of infima and to kill off critical triples.

Let us now examine the process needed to satisfy the requirement corresponding to $f \not\leq e$. To start, we will have to designate a number z targeted for A_f, and appoint traces x targeted for A_{p_0} or A_{w_1} and y targeted for A_{p_1} or A_{w_0} (to satisfy the two join requirements). The restriction imposed by incomparability requirements which prevents the placement of numbers into A_e forces the targets of the original traces x and y to be A_{p_0} and A_{p_1}, respectively. Now $p_0 < u_0, q_0$; $p_1 < u_1, q_1$; and neither p_0 nor p_1 is $\leq v_0$ or $\leq v_1$; thus the dangerous interval restrictions for the meet requirements corresponding to $q_0 \wedge u_1 \leq v_1$ and $q_1 \wedge u_0 \leq v_0$ force the entry of numbers into A_{p_0} and A_{p_1} to be separated by the appointment of a replacement trace for the first of the numbers to enter. If the replacement trace is targeted for A_{p_0} or A_{p_1}, then we have made no progress, as we have reestablished the initial situation. Thus, by symmetry, we may assume that there is a stage of the construction such that a number enters A_{p_0}, and a replacement trace, x_1, is appointed to enter A_{w_1} while a currently appointed trace, y_1, is still targeted to enter A_{p_1}.

Now $w_1 < q_0$; $p_1 < \tilde{p}$; and neither p_1 nor w_1 is $\leq p_0$; thus the dangerous interval restrictions for the meet requirements corresponding to $q_0 \wedge \tilde{p} \leq p_0$ forces the entry of numbers into A_{p_1} and A_{w_1} to be separated by the appointment of a replacement trace for the first of the numbers to enter. If x_1 enters first, then the appointment of a replacement trace to enter its target set before y_1 enters its target set will require that the replacement trace be targeted for p_0 or w_1; and a careful analysis shows that this will reestablish one of the situations which has already been discussed, so no progress will have been made. And if y_1 enters first, we will have violated the dangerous interval restriction corresponding to the meet requirement for $q_0 \wedge u_1 \leq v_1$. Thus we are faced with an insurmountable obstacle when trying to carry out a pinball machine construction.

The above argument can be formalized to show that L_{20} does not satisfy the embeddability condition EC. We will not do so, as we will prove a stronger result beginning in the next section; in particular, we will prove the following:

Theorem 1.2. *L_{20} cannot be embedded into the enumerable Turing degrees.*

2. Non-embeddability of L_{20}. Fix an usl embedding of L_{20} into $\mathbf{E}$. For each of the elements of L_{20} with notation as in Figure 1, we use the corresponding upper case letter to represent an enumerable set of the degree corresponding to the image of the lower case letter under the embedding. Without loss of generality, we may assume that these enumerable sets are chosen so that each such set G corresponding to the element g of L_{20} is expressed as a disjoint sum of sets of the degrees corresponding to the join irreducible elements of L_{20} which are $\leq g$. We will show that this embedding cannot be a lattice embedding. In fact, the only join and meet relations needed to prove the non-embeddability of L_{20} as a partial lattice are the following:

(2.1) $p_0 \vee w_1 \geq f$.

(2.2) $p_1 \vee w_0 \geq f$.

(2.3) $q_0 \wedge u_1 \leq v_1$.

(2.4) $q_1 \wedge u_0 \leq v_0$.

(2.5) $q_0 \wedge \tilde{p} \leq p_0$.

(2.6) $q_1 \wedge \tilde{p} \leq p_1$.

As we have fixed an usl embedding, (2.1) and (2.2) must hold. Thus we may fix computable functionals Ω_0 and Ω_1 such that $\Omega_0(P_0 \oplus W_1) = F$ and $\Omega_1(P_1 \oplus W_0) = F$. By speeding up the enumerations, we may assume that for each $x \leq s \in \mathcal{N}$, $\Omega_0^s(P_0^s \oplus W_1^s; x) \downarrow= F(x)$ and $\Omega_1^s(P_1^s \oplus W_0^s; x) \downarrow= F(x)$. Furthermore, as $\Omega_0^s(P_0^s \oplus W_1^s; x)$ and $\Omega_1^s(P_1^s \oplus W_0^s; x)$ are total, we may assume without loss of generality that for each $i \leq 1$ and all numbers s, n:

(2.7) If there is an injury to $\Omega_i(P_i \oplus W_{1-i}; n)$ at stage s, then $\omega_i(n,s) \geq \omega_{1-i}(n,s)$.

Define $\omega(n,s) = \max\{\omega_0(n,s), \omega_1(n,s)\}$.

In an attempt to contradict (2.5) or (2.6), we build enumerable sets D_0 and D_1 and computable functionals Δ_0, Δ_1, Λ_0, and Λ_1 which satisfy the following global requirement:

$$R: \ \forall i \leq 1 (D_i \ = \Delta_i(Q_i) \ = \ \Lambda_i(\tilde{P})).$$

(2.5) or (2.6) will be contradicted if we can show that for some $i \in \{0,1\}$,

(2.8) $\forall j \in \mathcal{N}(D_i \neq \tilde{\Psi}_j(P_i))$,

where $\{\tilde{\Psi}_j : \ j \in \mathcal{N}\}$ is an effective enumeration of all computable partial functionals. We may, however, fail to satisfy (2.8); thus predicated on the failure of (2.8) for $i = 0,1$ as witnessed by the functionals Ψ_0 and Ψ_1, respectively, we build enumerable sets $C_i^{\Psi_0,\Psi_1}$ and computable partial functionals $\Gamma_i^{\Psi_0,\Psi_1}$ and $\Xi_i^{\Psi_0,\Psi_1}$ for $i = 0,1$ (we omit the functional superscript when these will be clear from context), ensuring that the following requirement is satisfied:

$$R_{\vec{\Psi}} = R_{\Psi_0,\Psi_1} : \ \forall i \leq 1 (D_i = \Psi_i(P_i)) \ \rightarrow \ \forall i \leq 1 (C_i = \Gamma_i(Q_i) = \Xi_i(U_{1-i})).$$

(2.3) or (2.4) will be contradicted if we can show that for some $i \in \{0,1\}$,

(2.9) $\forall j \in \mathcal{N}(C_i \neq \tilde{\Phi}_j(V_{1-i}))$,

where $\{\tilde{\Phi}_j : \ j \in \mathcal{N}\}$ is an effective enumeration of all computable partial functionals. However, we may not succeed in satisfying (2.9); thus predicated on the failure of (2.9) for $i = 0,1$ as witnessed by the functionals Φ_0 and Φ_1, respectively, we build a computable partial functional $\Theta^{\Psi_0,\Psi_1,\Phi_0,\Phi_1}$ (we omit the functional superscript when these will be clear from context), ensuring that the following requirement is satisfied:

$$R_{\Psi_0,\Psi_1,\Phi_0,\Phi_1} : \ \forall i \leq 1 (D_i = \Psi_i(P_i)) \ \& \ \forall i \leq 1 (C_i = \Phi_i(V_{1-i})) \ \rightarrow \ F = \Theta(E).$$

This clearly allows us to conclude that L_{20} cannot be embedded into $\mathbf{E}$, as it contradicts an incomparability relationship of L_{20}. For compactness of notation, we will write $R_{\vec{\Psi},\vec{\Phi}}$ in place of $R_{\Psi_0,\Psi_1,\Phi_0,\Phi_1}$.

The intuition behind the plan to satisfy the requirements is as follows. We will work in the reverse order in which the requirements were presented. Thus we begin with an attempt to extend the domain of Θ, and argue that this attempt succeeds unless we succeed in diagonalizing against Φ_i or Ψ_i for some $i \leq 1$. We then argue that if we have an opportunity to diagonalize against Ψ_i which evaporates when a small number enters P_i, then we are presented with an opportunity to diagonalize against Φ_{1-i}. And if we have an opportunity to diagonalize against Φ_{1-i} which evaporates when a small number enters V_i, then Θ is corrected, and we can start anew without injury to any requirement; furthermore, this can only happen finitely often as $\Omega_i(P_i \oplus W_{1-i})$ is total for $i = 0, 1$. More specifically, we follow the alternating injuries to $\Omega_0(P_0 \oplus W_1)$ and $\Omega_1(P_1 \oplus W_0)$, showing that the first injury to $\Omega_i(P_i \oplus W_{1-i})$ in such an alternation is a P_i-injury. Between alternations, there may be multiple injuries to $\Omega_i(P_i \oplus W_{1-i})$; we show that each such injury can eventually be attributed to a change in the P_i-oracle, else we will be presented with a diagonalization opportunity. Now the entry of a number into F requires a simultaneous injury to $\Omega_0(P_0 \oplus W_1)$ and $\Omega_1(P_1 \oplus W_0)$; thus if we fail to have a diagonalization opportunity, E must also change, allowing us to correct $\Theta(E)$.

3. The Construction. We begin with the steps of the construction designed to satisfy R. At the end of stage s of the construction, for each integer $z \leq s$ and $i \leq 1$, we define axioms

(3.1) $\Delta_i^{s+1}(Q_i^s; z) = D_i^{s+1}(z)$ and

(3.2) $\Lambda_i^{s+1}(\tilde{P}^s; z) = D_i^{s+1}(z)$

whenever such an axiom is compatible with previously defined axioms. We require the use functions for the axioms specified in (3.1) to satisfy

(3.3) $\delta_i(z, s+1) = \omega_i(z, s)$.

(The ability to satisfy (3.3) for $i = 0, 1$ at all stages s follows from the fact that $q_i \geq p_i, w_{1-i}$.) We require the use functions for the axioms specified in (3.2) to satisfy

(3.4) $\lambda_i(y_i, s+1) = \omega(y_i, s)$

whenever there are functionals Ψ_j and Φ_j for $j = 0, 1$ and numbers x_0, x_1, y_0, and y_1 such that an attack on $R_{\vec{\Psi},\vec{\Phi}}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ has been begun by stage s but has not been cancelled by stage s. (We will not be able to ensure that (3.4) holds at all stages s, but will satisfy this condition whenever changes in the $\tilde{P}$-oracle allow us to redefine such an axiom. The satisfaction of (3.4) at the appropriate stages will allow us to diagonalize against Ψ_i for $i = 0, 1$.) We track the progress of the construction by defining a function

$$\ell_{\Delta_i,\Lambda_i,\Psi_i}(s) = \max\{x : \forall z \leq x(\Delta_i^s(Q_i^s; z) \downarrow = \Lambda_i^s(\tilde{P}^s; z) \downarrow = \Psi_i^s(P_i^s; z) \downarrow)\}$$

which measures the common length of agreement of $\Delta_i(Q_i)$, $\Lambda_i(\tilde{P})$, and $\Psi_i(P_i)$.

Given functionals Ψ_0 and Ψ_1, we employ the following strategy to satisfy $R_{\vec{\Psi}}$. At the end of stage s of the construction, for each $i \leq 1$ and $z \leq s$, we define axioms

(3.5) $\Gamma_i^{s+1}(Q_i^s; z) = C_i^{s+1}(z)$, and

(3.6) $\Xi_i^{s+1}(U_{1-i}^s; z) = C_i^{s+1}(z)$

whenever such axioms are compatible with previously defined axioms, unless there exist $j, k \leq 1$ and numbers x_0, x_1, y_0, y_1, and n such that $x_0 \leq z$ or $x_1 \leq z$ and an attack on $R_{\vec{\Psi},\vec{\Phi}}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ which is not yet cancelled is currently *j-suspended* (this term will be defined during the construction; it denotes a stage at which we are waiting for an injury to a certain computation, which will allow us to resume the attack), or $\Psi_j^s(P_j^s; y_j)$ is undefined. We require the use functions for the axioms newly specified in (3.5) to satisfy

(3.7) If $\Gamma_i^{s+1}(Q_i^s; z)$ is defined then $\gamma_i(z, s+1) = \omega_i(z, s)$.

(The ability to satisfy (3.7) for $i = 0, 1$ at all stages s follows from the fact that $q_i \geq p_i, w_{1-i}$.) We require the use functions for the axioms newly specified in (3.6) to satisfy

(3.8) If $\Xi_i^{s+1}(U_{1-i}^s; x_i)$ is defined then $\xi_i(x_i, s+1) \geq \psi_{1-i}(y_{1-i}, s)$ and

(3.9) If $\Xi_i^{s+1}(U_{1-i}^s; x_i)$ is defined then $\xi_i(x_i, s+1) \geq \omega(n, s)$.

for $i = 0, 1$, whenever numbers x_0, x_1, y_0, and y_1 and functionals Φ_0 and Φ_1 are specified such that an attack on $R_{\vec{\Psi},\vec{\Phi}}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ has been begun by stage s but has not been cancelled by or at stage s. (The value of y_{1-i} will be reset only at a stage s at which $\Xi_i^{s+1}(U_{1-i}^s; x_i)$ is undefined, so the ability to satisfy (3.8) at all stages s will follow from the fact that $p_{1-i} \leq u_{1-i}$ for $i = 0, 1$. We will not be able to ensure that (3.9) holds at all stages s, but will satisfy this condition whenever changes in the U_{1-i}-oracle allow us to redefine such an axiom. The satisfaction of (3.7)-(3.9) at the appropriate stages will allow us to diagonalize against Φ_i for $i = 0, 1$.) We track the progress of the construction by defining a function

$$\ell_{\Gamma_i,\Xi_i,\Phi_i}(s) = \max\{x : \forall z \leq x(\Gamma_i^s(Q_i^s; z)\downarrow = \Xi_i^s(U_{1-i}^s; z)\downarrow = \Phi_i^s(V_{1-i}^s; z)\downarrow)\}$$

which measures the common length of agreement between $\Gamma_i(Q_i)$, $\Xi_i(U_{1-i})$, and $\Phi_i(V_{1-i})$. The values chosen for the use functions being defined are always the smallest numbers which are consistent both with the above requirements placed on use functions, and the requirement that use functions be non-decreasing on each argument.

Fix functionals Ψ_0, Ψ_1, Φ_0, and Φ_1, and let $\hat{R} = R_{\vec{\Psi},\vec{\Phi}}$. We effectively partition the integers into infinitely many infinite sets, and effectively assign a different such set to each requirement $\hat{R}$. Let the correspondence assign the set $\hat{S} = S_{\Psi_0,\Psi_1,\Phi_0,\Phi_1}$ to $\hat{R}$.

We try to satisfy $\hat{R}$ as follows. We cycle through the numbers $n \leq s$ at stage s, trying to define $\Theta(E; n)$. Thus for each $n \leq s$, we follow the sequence of steps below. We begin a new attack on $\hat{R}$ for n at stage s whenever the conditions for Step 1 are satisfied. These conditions provide the opportunity to extend the definition of $\Theta(E)$ to a new argument. Action to extend the definition is taken in Step 2. Step 3 governs the type of action to be taken (in Step 4) for each attack on $\hat{R}$ which is now in progress (so has not yet been cancelled). Fix n. We want to act when there is an injury to $\Omega_i(P_i \oplus W_{1-i}; n)$ at stage s for some $i \leq 1$. (We will define a stage to be an *i-stage* if there is an injury to $\Omega_i(P_i \oplus W_{1-i}; n)$ and the previous injury to either Ω was to $\Omega_{1-i}(P_{1-i} \oplus W_i; n)$.) Such attacks may occur many times, so we let i_j be the i which determines the jth attack (we begin with i_0), and require that the value of i_j alternate between 0 and 1 for successive attacks. We let s_j be the stage at which we begin to the attempt for i_j. The injury at stage s_j provides an opportunity to try to satisfy $\hat{R}$.

The action, in Step 4, will depend on the nature of the injury to the computation for the functional Ω_{i_j} at stage s_j. *Successful* attacks for n will be those which have reached a stage which forces the opponent to correct the computation of $\Omega_i(P_i \oplus W_{1-i}; n)$ for some $i \leq 1$ in order to prevent us from winning the requirement outright. Such action will allow us to correct $\Theta(E; n)$. The opponent can only act this way finitely often for each n, so if we restart new attacks each time such action is taken, we will argue that we eventually succeed in satisfying the requirement for some n. The action cases of Step 4 reflect the following situations. In Case 1, we have simultaneous permission to diagonalize by putting a number into C_{i_j}, and the attack will be successful. Otherwise, we follow Case 2, and have simultaneous permission to diagonalize by putting a number into D_{i_j}. This will provide a win for the requirement unless the opponent takes action. The type of action gives rise to three subcases.

In Subcase 2.1, the opponent places a small number into $P_{\hat{\imath}_j}$. This provides us with delayed permission to diagonalize by putting a number into $D_{\hat{\imath}_j}$. We declare success here while suspending the attack. The opponent can only counter by putting a smaller number into $P_{\hat{\imath}_j}$ which provides delayed permission to place a number into C_{i_j}, and so we see that success is assured.

In Subcase 2.2, the opponent places a small number into W_{i_j}, providing delayed permission to diagonalize by placing a number into $C_{\hat{\imath}_j}$, and so declare the attack to be successful.

In Subcase 2.3, the opponent will choose just to provide new expansionary stages for a meet requirement, without making progress towards preventing us from computing $\Theta(E) = F$ by having the opponent diagonalize this computation at n. In this situation, we can lift up our traces and begin a *replacement attack* with the larger traces, allowing us to make use of larger changes in sets computed by the opponent for the sake of diagonalization. Replacement attacks which are not separated by cancellation caused by an element $\leq \theta(n)$ entering E are tied to fixed arguments of Ω_0 and Ω_1, so only finitely many can occur. Thus we will eventually be forced into a different case or subcase.

Step 1: We wait for a stage $s_0 \geq n$ at which no prior uncancelled attack on $\hat{R}$ is successful or i-suspended for any $i \leq 1$, and

(3.10) $\Theta^{s_0}(E^{s_0-1})[n-1] = F^{s_0-1}[n-1]$ and $\Theta^{s_0}(E^{s_0-1}; n)$ is undefined,

and for some choice of $x_0, x_1, y_0, y_1 \in \hat{S}$ such that $x_i \notin C_i^{s_0}$, $y_i \notin D_i^{s_0}$ and such that there are no prior attacks on $\hat{R}$ using numbers $\geq \min\{x_0, x_1, y_0, y_1\}$, we have

(3.11) $x_0, x_1, y_0, y_1 \geq n, \omega(n, s_0)$;

(3.12) $x_i \leq \ell_{\Gamma_i,\Xi_i,\Phi_i}(s_0)$, $y_i \leq \ell_{\Delta_i,\Lambda_i,\Psi_i}(s_0)$ for $i = 0, 1$; and

(3.13) $\xi_i(x_i, s_0) \geq \psi_{1-i}(y_{1-i}, s_0)$ for $i = 0, 1$.

Should s_0 be found, we fix the first (in a specified effective ordering) such quadruple $\langle x_0, x_1, y_0, y_1\rangle$ and *begin an attack on* $\hat{R}$ *for* n *at* s_0 *through* $\langle x_0, x_1, y_0, y_1\rangle$. We define s_0 to be the *initial stage* for this attack. This attack will be immediately *cancelled* at stage $t > s_0$ if there is an injury to $\Theta(E, n)$ at some stage $r \in (s_0, t]$.

Step 2: Define $\Theta^{s_0+1}(E^{s_0}; n) = F^{s_0}(n)$. The use of this computation is $\theta(n, s_0+1) = \max\{\phi_0(x_0, s_0), \phi_1(x_1, s_0)\}$. Set $t_0 = s_0$. s_0 is said to be both a 0-*stage* and a 1-*stage* for n. Now go to Step 3, with $j = 1$.

Step 3: Suppose that either $j = 1$, or that $j > 1$ and s_{j-1}, t_{j-1}, i_{j-1}, and $\hat{\imath}_{j-1}$ are defined. We search for the first stage $s > t_{j-1}$ and number $i \leq 1$ such that $i = \hat{\imath}_{j-1}$ if $j > 1$ and either

(3.14) $W^s_{1-i}[\omega_i(n, t_{j-1} + 1)] \neq W^{s-1}_{1-i}[\omega_i(n, t_{j-1} + 1)]$, or

(3.15) $P^s_i[\omega_i(n, t_{j-1} + 1)] \neq P^{s-1}_i[\omega_i(n, t_{j-1} + 1)]$.

If s and i satisfying (3.14) or (3.15) exist, we let s_j be the least such s, and we let i_j be the least such i for s_j if $j = 1$ and $i_j = \hat{\imath}_{j-1}$ if $j > 1$; in both cases, we define $\hat{\imath}_j = 1 - i_j$. s_j is called an i_j-*stage*. We now go to Step 4.

Step 4: We follow the instructions of the first case which applies. (Attacks will be declared to be *successful* below when we will later be able to show that the action taken for the current attack on $\hat{R}$ for n ensures that either a hypothesis of $\hat{R}$ will not be satisfied, or the attack will later be cancelled because of an injury to $\Theta(E; n)$.)

Case 1: (3.14) holds at stage s_j. Put x_{i_j} into $C^{s_j+1}_{i_j}$ and declare the current attack on $\hat{R}$ for n to be *successful* at stage s_j. No further action will be taken for any attack on $\hat{R}$ for n until (if ever) this attack is cancelled.

Case 2: (3.15) holds at stage s_j. Put y_{i_j} into $D^{s_j+1}_{i_j}$. The attack becomes i_j-*suspended* at stage s_j. We now wait for the first stage $t \geq s_j$ such that there is a number $\tilde{y} \in \hat{S}$ which is larger than any number used earlier in the construction for $\hat{R}$ and the inequalities $\tilde{y} \leq \ell_{\Delta_{i_j}, \Lambda_{i_j}, \Psi_{i_j}}(t)$ and $y_{\hat{\imath}_j} \leq \ell_{\Delta_{\hat{\imath}_j}, \Lambda_{i_j}, \Psi_{\hat{\imath}_j}}(t)$ are satisfied. If t is found and the i_j-suspension has not been lifted prior to stage t, then the i_j-*suspension is lifted* at stage t and we set $t_j = t$. If, in the course of the search for t_j, we encounter a stage $r \geq s_j$ at which one of (3.16) or (3.17) below holds (this requires that $r \leq t_j$ should t_j exist), we fix the first such r and adopt the first of Subcases 2.1 or 2.2 which applies; otherwise we adopt Subcase 2.3.

(3.16) $P^r_{\hat{\imath}_j}[\omega_{\hat{\imath}_j}(n, s_j - 1)] \neq P^{s_j-1}_{\hat{\imath}_j}[\omega_{\hat{\imath}_j}(n, s_j - 1)]$.

(3.17) $W^r_{i_j}[\omega_{\hat{\imath}_j}(n, s_j - 1)] \neq W^{s_j-1}_{i_j}[\omega_{\hat{\imath}_j}(n, s_j - 1)]$.

Subcase 2.1: (3.16) holds. Put $y_{\hat{\imath}_j}$ into $D^{r+1}_{\hat{\imath}_j}$ and declare the current attack on $\hat{R}$ for n to be *successful* at stage r. No further action will be taken for any attack on $\hat{R}$ for n until (if ever) this attack is cancelled, except for the action specified to complete this subcase. This attack becomes $\hat{\imath}_j$-*suspended* at stage r. The $\hat{\imath}_j$-*suspension* will be lifted at the first stage $s \geq r$ for which there is an injury to

$\Psi_{\hat{\imath}_j}(P_{\hat{\imath}_j}; y_{\hat{\imath}_j})$ at some stage in $[r, s]$ and $\Psi^s_{\hat{\imath}_j}(P^s_{\hat{\imath}_j}; y_{\hat{\imath}_j})$ is defined. At the stage s at which the $\hat{\imath}_j$-suspension is lifted, put x_{i_j} into $C^{s+1}_{i_j}$.

Subcase 2.2: (3.17) holds. Put $x_{\hat{\imath}_j}$ into $C^{r+1}_{\hat{\imath}_j}$. Declare the current attack on $\hat{R}$ for n to be *successful* at stage r. No further action will be taken for any attack on $\hat{R}$ for n until (if ever) this attack is cancelled.

Subcase 2.3: Otherwise. The attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ is *cancelled* at stage t_j, we set $\tilde{y}_{\hat{\imath}_j} = y_{\hat{\imath}_j}$ and begin a *replacement attack* on $\hat{R}$ for n through $\langle x_0, x_1, \tilde{y}_0, \tilde{y}_1\rangle$. We now return to Step 3, replacing j with $j + 1$.

This completes the construction. We will show in the next section that the construction ensures the satisfaction of all requirements.

4. The Proof. We now complete the proof, showing that L_{20} cannot be embedded into the enumerable Turing degrees. The theorem follows immediately from the satisfaction of all requirements, so our goal will be to show that this is the case. Fix functionals Ψ_0, Ψ_1, Φ_0, and Φ_1, and set $\hat{R} = R_{\vec{\Psi},\vec{\Phi}}$. We begin with a lemma specifying relationships between use functions.

Lemma 4.1. *Suppose that $s \geq n$ and $i \leq 1$. Then (3.3) and (3.8) hold, as does*

(4.1) $\gamma_i(x_i, s) \geq \omega_i(n, s)$.

(We assume here that for any such inequality in which x_i or y_i is mentioned, there is an uncancelled attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$.)

Proof. We first show that (3.3) holds. By the construction, if t is any stage at which a new axiom $\Delta^{t+1}_i(Q^t_i; n)$ is defined, then the construction requires (3.3) to hold. As $p_i, w_{1-i} \leq q_i$, if r is any stage at which an axiom $\Omega_i(P_i \oplus W_{1-i}; n)$ is injured (and so a new axiom is defined), then the axiom $\Delta_i(Q_i; n)$ is also injured at stage r, so a new axiom $\Delta^{r+1}_i(Q^r_i; n)$ is defined to satisfy (3.3).

A proof similar to that in the preceding paragraph shows that (4.1) holds at s; the only non-notational difference is that when a new axiom $\Omega^r_i(P^r_i \oplus W^r_{1-i}; n)$ is defined, there may be a delay before we reach the first stage $s \geq r$ at which we define a new axiom $\Gamma^{s+1}_i(Q^s_i; n)$. We now note that by (3.11), $n \leq x_i$, so as the use function $\gamma_i(m, t)$ is non-decreasing in m, $\gamma_i(x_i, t) \geq \omega_i(n, t)$ whenever a corresponding x_i is defined. (4.1) now follows.

It remains to verify (3.8). By the construction, if t is any stage at which a new axiom $\Xi^{t+1}_i(U^t_{1-i}; x_i)$ is defined, then the quadruple $\langle x_0, x_1, y_0, y_1\rangle$ has been specified, and the construction requires (3.8) to hold. As $\xi_i(x_i, s)$ is a use function, it is non-decreasing in s. The construction may change its choice of number for x_i from y_{1-i} to $\tilde{y}_{1-i}$ at stage t only if Step 4, Case 2.3 of the construction is followed at some stage $s \leq t$, in which case there is an injury to $\Psi_{1-i}(P_{1-i}; y_{1-i})$ at some stage $r \in [s, t]$, the attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ becomes $(1-i)$-suspended at stage r, and this suspension is not lifted until $\tilde{y}_{1-i}$ is selected to replace y_{1-i}. As $p_k \leq u_k$ for $k = 0, 1$, either $\Xi^{r+1}_i(U^r_{1-i}; x_i)$ is undefined (in which case we set $u = r$), or there is an injury to $\Xi_i(U_{1-i}; x_i)$ at some stage in $u \in [r, t]$. Furthermore, $\Xi^{v+1}_i(U^v_{1-i}; x_i)$ is undefined at all stages $v \in [u, t)$ because of this $(1-i)$-suspension. (3.8) now follows. $\square$

The next lemma specifies use-function inequalities which will enable us to show that we will eventually be able to satisfy R, $R_{\vec{\Psi}}$, and $\hat{R}$. The first clause tells us that we can always attribute the change in value of ω_{i_j} at stage s_j to a change in P_{i_j} (rather than W_{1-i_j}), else we will satisfy the requirement based on the change at s_j. This is used, in the second clause, to establish an inequality relating λ_k and ω uses. To this end, we fix the quadruple $\langle x_0, x_1, y_0, y_1\rangle$ through which the attack on $\hat{R}$ for n is begun at stage s_0.

Lemma 4.2. *Fix $j > 0$ such that the sequence of attacks on $\hat{R}$ for n begun at s_0 has not been declared to be successful at any stage $t \leq s_j$, and assume that no attack in this sequence is ever cancelled due to Θ-injury. Then:*

(i) $P_{i_j}^{s_j}[\omega_{i_j}(n, s_j)] \neq P_{i_j}^{s_j-1}[\omega_{i_j}(n, s_j)]$ and $W_{\hat{\imath}_j}^{s_j}[\omega_{i_j}(n, s_j)] = W_{\hat{\imath}_j}^{s_j-1}[\omega_{i_j}(n, s_j)]$ if $j > 0$.

(ii) $\lambda_k(y_k, s_j + 1) \geq \omega(n, s_j)$ for $k = 0, 1$.

Proof. The lemma follows by the cancellation feature and by (3.11) for $j = 1$. We now proceed, case by case, by induction on $j > 0$.

(i): The definition of s_j implies that either $P_{i_j}^{s_j}[\omega_{i_j}(n, s_j)] \neq P_{i_j}^{s_j-1}[\omega_{i_j}(n, s_j)]$ or else $W_{\hat{\imath}_j}^{s_j}[\omega_{i_j}(n, s_j)] \neq W_{\hat{\imath}_j}^{s_j-1}[\omega_{i_j}(n, s_j)]$. Should the latter hold, then we would follow Step 4, Case 1 of the construction at stage s_j, and would declare the current attack on $\hat{R}$ for n to be successful at stage s_j, contrary to hypothesis.

(ii): By induction, $\lambda_k(y_k, s_{j-1} + 1) \geq \omega(n, s_{j-1}) \geq \omega_{i_j}(n, s_{j-1})$. By the construction, s_{j-1} is a $\hat{\imath}_j$-stage. Hence as, by (3.14)-(3.17), there is no injury to $\Omega_{i_j}(P_{i_j} \oplus W_{\hat{\imath}_j}; n)$ at any stage in $[s_{j-1}, s_j)$ and as use functions are increasing on each argument, $\lambda_k(y_k, s_j) \geq \omega_{i_j}(n, s_j - 1)$. By (i), there is a P_{i_j}-injury to $\Omega_{i_j}(P_{i_j} \oplus W_{\hat{\imath}_j}; n)$ at stage s_j, so as $p_{i_j} < \tilde{p}$, there is an injury to $\Lambda_k(\tilde{P}; y_i)$ at stage s_j, allowing us to satisfy (ii). □

Lemma 4.3. *R is satisfied.*

Proof. As $\Omega_i(P_i \oplus W_{1-i})$ is total for $i = 0, 1$, we have that $\lim_s \omega(x, s)$ exists for all x. By (3.3) and (3.4), $\delta_i(n, s+1) = \omega_i(n, s)$ and $\lambda_i(y_i, s+1) = \omega(y_i, s)$ whenever the use functions on the left hand side are newly defined; and the construction requires $\delta_i(n, s+1)$ is defined for all $s \geq n$ and $\lambda_i(y_i, s+1)$ is defined for all $s \geq y_i$. Thus $\Delta_i(Q_i)$ and $\Lambda_i(\tilde{P})$ are total.

An examination of the construction now shows that we place a number y_i into D_i^{s+1} for $i = 0, 1$ only at a stage s at which there are n, x_0, x_1, and y_{1-i} such that an attack on $\hat{R}$ is in progress through $\langle x_0, x_1, y_0, y_1\rangle$ and there is a P_i-injury to $\Omega_i(P_i \oplus W_{1-i}; n)$ at stage s. By Lemma 4.1, (3.3) holds so $\delta_i(n, s) = \omega_i(n, s-1)$. We recall that r was chosen in Step 4, Subcase 2.2 of the construction to be the first stage $\geq s_j$ at which (3.16) or (3.17) holds; thus by Lemma 4.2(ii), $\lambda_i(y_i, s+1) \geq \omega_i(n, s)$. Now either $\lambda_i(y_i, s) = \lambda_i(y_i, s+1) \geq \omega_i(n, s) \geq \omega_i(n, s-1)$, or there is an injury to $\Lambda_i(\tilde{P}; y_i)$ at stage s; and in the first case, as $p_i < \tilde{p}$, the P_i-injury to $\Omega_i(P_i \oplus W_{1-i}; n)$ at stage s will cause $\Lambda_i(\tilde{P}; y_i)$ to be injured at stage s. Thus we define new axioms $\Delta_i^{s+1}(Q_i^s; n) = D^{s+1}(y_i)$ and $\Lambda_i^{s+1}(\tilde{P}^s; y_i) = D^{s+1}(y_i)$ at the end of stage s, so R is satisfied. □

If $D_i \neq \Psi_i(P_i)$ for some $i \leq 1$, then $R_{\vec{\Psi}}$ and $\hat{R}$ are satisfied. So we assume that $D_i = \Psi_i(P_i)$ for $i \leq 1$. By Lemma 4.3 and this assumption, for $i = 0, 1$, (4.2) $\lim_s \ell_{\Delta_i,\Lambda_i,\Psi_i}(s) = \infty$.

Under this assumption, we show that suspensions are always lifted or cancelled.

Lemma 4.4. *Fix n. Suppose that we are given an attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ which is i-suspended at stage s for some $i \le 1$. Then there is a least $t > s$ at which the suspension is lifted or the attack is cancelled. Furthermore, there is a stage r such that for every $t \ge r$, no uncancelled attack on $\hat{R}$ for n is i-suspended for any $i \le 1$.*

Proof. An attack on $\hat{R}$ for n can only be i-suspended in Case 2 of Step 4 of the construction. Fix $i \le 1$ such that the attack is i-suspended at stage s.

As $\Delta_i^{r+1}(Q_i^r)$ and $\Lambda_i^{r+1}(\tilde{P}^r)$ are compatible with D_i^{r+1} for all r and $\Delta_i^{r+1}(Q_i^r; y_i)$ and $\Lambda_i^{r+1}(\tilde{P}^r; y_i)$ are defined for all $r \ge y_i$, it follows from the construction and (3.12) that $y_i \in D_i^{s+1} - D_i^s$ and $\Psi_i^s(P^s; y_i) = 0$.

First assume that the i-suspension occurs at the beginning of Step 4, Case 2 of the construction. Fix a number $\tilde{y}_i \in \hat{S}$ which has not yet been used in the construction and which satisfies (3.11). By the construction, no new attacks on $\hat{R}$ for any $m \ge n$ will begin until the current attack is cancelled or the i-suspension is lifted. By (4.2), there must be a first $t > s$ such that (3.12) holds for $\tilde{y}_i$ and $\tilde{y}_{1-i}$. The construction now lifts the i-suspension of the original attack at stage t and also cancels this attack, if the attack has not been cancelled earlier.

Now assume that the i-suspension occurs during Step 4, Subcase 2.1 of the construction. By the construction, no new attacks on $\hat{R}$ for any $m \ge n$ will begin until the current attack is cancelled or the i-suspension is lifted. Hence by (4.2), there must be a first $t > s$ such that there is an injury to $\Psi_i(P_i; y_i)$ at stage t and (3.12) again holds for y_i. The construction now lifts the i-suspension at stage t if the attack was not cancelled earlier.

An attack on $\hat{R}$ for n can be newly i-suspended at stage r only if there is an $s_j < r$ as in the construction. Furthermore, at most one such suspension can begin before a stage t_j as in the construction is found (if ever). And there is no such suspension after stage t_j unless s_{j+1} is defined, in which case there is no such suspension in the interval (t_j, s_{j+1}). By the definition of s_j, there is an injury to $\Omega_k(P_k \oplus W_{1-k})$ at stage s_j for some $k \le 1$. So as $\Omega_k(P_k \oplus W_{1-k})$ is total for $k = 0, 1$, the construction will only find finitely many stages s_j corresponding to n. The last part of the lemma now follows from the first part. $\square$

In the next lemma, we prove inequalities involving the use functions ξ_k for $k = 0, 1$. These are needed to show that $R_{\vec{\Psi}}$ is satisfied.

Lemma 4.5. *Suppose that there is an attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1\rangle$ at stage s_j which has not been declared to be successful by stage $s_j - 1$. Then for $k = i_j$ and $j > 0$, if $\xi_k(x_k, s_j)$ is defined then $\xi_k(x_k, s_j) \ge \omega_k(n, s_j - 1)$.*

Proof. As $\Omega_k(P_k \oplus W_{1-k}; x_k)$ is not injured at any stage $t \in (s_0, s_1)$, the lemma follows from (3.9) for $j = 1$.

Suppose that $j > 1$. We note that s_{j-1} is a $(1-k)$-stage. As the attack on $\hat{R}$ for n at stage s_{j-1} was not successful, Step 4, Subcase 2.3 of the construction must have been followed at that stage, so y_{1-k} was placed into $D_{1-k}^{s_{j-1}+1}$ at the beginning of Step 4, Case 2. The existence of s_j implies the existence of t_{j-1}; so as all axioms $\Psi_{1-k}^{t+1}(P_{1-k}^t; y_{1-k}) = m$ declared at stages $t < s_{j-1}$ set $m = 0 = D_{1-k}^{t+1}(y_{1-k})$, it follows from (2.12) that there is a stage $r \in [s_{j-1}, t_{j-1}]$ at which there is an injury to $\Psi_{1-k}(P_{1-k}; y_{1-k})$. As $p_{1-k} < u_{1-k}$, it follows from Lemma 4.1 and (3.8) that there is a stage $r \in [s_{j-1}, t_{j-1}]$ at which there is an injury to $\Xi_k(U_{1-k}; x_{1-k})$. Thus we define a new axiom $\Xi_k^{t_{j-1}+1}(U_{1-k}^{t_{j-1}}; x_k) = 1$, and by (3.9), $\xi_k(x_k, t_{j-1} + 1) \geq \omega_k(n, t_{j-1})$. As there is no injury to $\Omega_k(P_k \oplus W_{1-k})$ at any stage in (t_{j-1}, s_j), the lemma now follows from the increasing property of use functions. □

Lemma 4.6. *$R_{\vec{\Psi}}$ is satisfied.*

Proof. Axioms for $\Gamma_i(Q_i; z)$ and $\Xi_i(U_{1-i}; z)$ are declared at each sufficiently large stage s at which there is no attack on $\hat{R}$ for any n through any $\langle x_0, x_1, y_0, y_1 \rangle$ which is i-suspended for some $i \leq 1$ and for which $\min\{x_0, x_1, y_0, y_1\} \leq z$, and $\Psi_k^s(P_k^s, y_k)$ is defined for $k = 0, 1$. Whenever a new number is selected for an attack on $\hat{R}$, it is larger than any numbers previously used in attacks on $\hat{R}$. Hence by (4.2) and Lemma 4.4, such axioms will be declared at all sufficiently large stages s. Now by Lemma 4.1, $\gamma_i(x_i, s+1) \leq \omega_i(x_i, s)$ for all stages s for which $\gamma_i(x_i, s+1)$ is defined, and $\Omega_i(P_i \oplus W_{1-i})$ is total for $i = 0, 1$. Hence $\Gamma_i(Q_i)$ is total. Furthermore, by the construction,

$$\xi_i(x_i, s+1) \leq \max\{\omega(x_i, s), \psi_{1-i}(y_{1-i}, s)\}$$

for all stages s for which $\xi_i(x_i, s+1$ is defined. A change from y_{1-i} to $\tilde{y}_{1-i}$ as the number corresponding to x_i must follow the suspension of an attack on $\hat{R}$ for n at a stage following the stage at which the correspondence of x_i to y_{1-i} was set. By Lemma 4.4, this can occur only finitely often. Hence there is a final y_{1-i} corresponding to x_i (whenever x_i is specified), which we fix. We again note that $\Omega_i(P_i \oplus W_{1-i})$ is total for $i = 0, 1$, and by Lemma 4.3, $\Psi_{1-i}(P_{1-i})$ is total for $i = 0, 1$. Hence $\Xi_i(U_{1-i})$ is total.

We now consider the stage at which a number x_i is placed into C_i for $i \leq 1$.

Case 1: s_i is an i-stage. It follows from Lemma 4.1 and the fact that $p_i, w_{1-i} < q_i$, that either $\Gamma_i^{s_j}(Q_i^{s_j-1}; x_i)$ is undefined, or there is an injury to $\Gamma_i(Q_i; x_i)$ at stage s_j. Thus at the next stage $s \geq s_j$ at which $\Gamma_i^{s+1}(Q_i^s; x_i)$ is defined, the axiom declared is compatible with $C_i^{s+1}(x_i)$. We now consider several subcases, depending on the case and subcase followed by the construction at stage s_j.

Subcase 1.1: The action is taken at a stage s_j at which Step 4, Case 1 of the construction is followed. By Lemma 4.5 and as $w_{1-i} < u_{1-i}$, either $\Xi_i^{s_j}(U_{1-i}^{s_j}; x_{1-i})$ is undefined, or there is an injury to $\Xi_i(U_{1-i}; x_i)$ at stage s_j. Thus at the next stage $s \geq s_j$ at which $\Xi_i^{s+1}(U_{1-i}^s; x_i)$ is defined, the axiom declared is compatible with $C_i^{s+1}(x_i)$.

Subcase 1.2: Suppose that Subcase 2.1 is followed at s_j. Then the attack begun at stage s_j is i-suspended at stage s_j. By Lemma 4.4 and (4.2), there is a first stage $t \geq s_j$ at which this suspension is lifted. Now at some stage $r \in [s_j, t]$, we place y_{1-i} into D_{1-i}^{r+1}. By (4.2) and as $\Psi_{1-i}^r(P_{1-i}^r; y_{1-i}) = 0$, there must be a later stage $\tilde{s} \in (r, t]$ at which $\Psi_{1-i}(P_{1-i}; y_{1-i})$ is injured. By Lemma 4.1, (3.8) holds at stage $\tilde{s}$ so as $p_{1-i} < u_{1-i}$, there will also be an injury to $\Xi_i(U_{1-i}; x_i)$ at stage

$\tilde{s}$, and $\Xi_i^{\tilde{s}+1}(U_{1-i}^{\tilde{s}}; x_i)$ is undefined for all $s \in [\tilde{s}, t)$. Thus we will be able to define $\Xi_i^{t+1}(U_{1-i}^t; x_i) = C_i^{t+1}(x_i)$.

Case 2: s_j is a $(1-i)$-stage. Then Subcase 2.2 must be followed at stage s_j, and the attack begun at stage s_j is i-suspended at stage s_j. Furthermore, we place y_{1-i} into $D_{1-i}^{s_j+1}$. By (4.2) and as $\Psi_{1-i}^{s_j}(P_{1-i}^{s_j}; y_{1-i}) = 0$, there must be a later stage $\tilde{s} \in (s_j, t]$ at which $\Psi_{1-i}(P_{1-i}; y_{1-i})$ is injured. By Lemma 4.1, (3.8) holds at stage $\tilde{s}$ so as $p_{1-i} < u_{1-i}$, there will also be an injury to $\Xi_i(U_{1-i}; x_i)$ at stage $\tilde{s}$, and $\Xi_i^{s+1}(U_{1-i}^s; x_i)$ is undefined for all $s \in [\tilde{s}, t)$. Thus we will be able to define $\Xi_i^{t+1}(U_{1-i}^t; x_i) = C_i^{t+1}(x_i)$.

Let r be the stage at which (3.17) holds for s_j. As $w_{1-i} < q_i$, so by Lemma 4.1(i), the W_{1-i}-injury to $\Omega_i(P_i \oplus W_{1-i}; n)$ at stage r forces an injury to $\Gamma_i(Q_i; x_i)$ at stage r. Thus we will be able to redefine $\Gamma_i^{t+1}(Q_i^t; x_i) = C_i^{t+1}(x_i)$. □

If $C_i \neq \Phi_i(V_{1-i})$ for some $i \leq 1$, then $\hat{R}$ is satisfied. So we assume that this is not the case. It then follows that for $i = 0, 1$, $\Psi_i(P_i)$ and $\Phi_i(V_{1-i})$ are total and

(4.3) $\lim_s \ell_{\Gamma_i, \Xi_i, \Phi_i}(s) = \infty$.

Lemma 4.7. *Suppose that $\Theta(E)$ and F are compatible. Then $\Theta(E)$ is total.*

Proof:. We proceed by induction on n, showing that $\Theta(E; n)$ is defined. Fix n, and assume that $\Theta(E; m)$ is defined for all $m < n$. This assumption and Lemma 4.4 allow us to fix a stage t_0 such that for all $t \geq t_0$ and $m < n$, $\Theta^{t+1}(E^t; m) = \Theta(E; m)$ and no attack on $\hat{R}$ for any $m < n$ is suspended at stage t. As $\Omega_0(P_0 \oplus W_1)$ and $\Omega_1(P_1 \oplus W_0)$ are total, we may fix a stage $t_1 \geq t_0$ such that for all $t \geq t_1$, $\omega(n, t) = \lim_s \omega(n, s) = \omega(n)$ and $E^t[\omega(n)] = E[\omega(n)]$; and as we have assumed that $\Phi_i(V_{1-i})$ is total for $i \leq 1$, we may assume that $\phi_i(x_i, t) \downarrow = \phi_i(x_i)$ for all $t \geq t_1$ and $i \leq 1$. We now note by definition that no attack on $\hat{R}$ for n is cancelled at any stage $t > t_1$. Thus if $\Theta^{t+1}(E^t; n) \downarrow$ for any $t > t_1$, then $\Theta^{s+1}(E^s; n) \downarrow = \Theta^{t+1}(E^t; n)$ for all $s \geq t$. The induction hypothesis will follow if we can show that $\Theta^{t+1}(E^t; n)$ is defined for some $t > t_1$.

If $\Theta^{t_1+1}(E^{t_1}; n)$ is defined, then we are done. Suppose not. It suffices to show that there is a $t > t_1$ at which an attack on $\hat{R}$ for n is in progress, as then, by Step 2 of the construction for the smallest such t, $\Theta^{t+1}(E^t; n)$ is defined. Note that as we require $\Theta^{t+1}(E^t; n)$ is defined in order to begin an attack on $\hat{R}$ for $m > n$, and as, if an attack on $\hat{R}$ for n at s is cancelled, then all attacks on $\hat{R}$ for $m > n$ at s are cancelled, there will be no attack on $\hat{R}$ at $t > t_1$ for any $m > n$ if there is no attack on $\hat{R}$ for n at stage t.

Fix $y_0, y_1 \in \hat{S}$ such that $y_0, y_1 > n, \omega(n, t_1)$, $y_i \notin C_i^{t_1+1}$ for $i \leq 1$, and y_0 and y_1 are greater than any numbers in any quadruple through which there was an attack on $\hat{R}$ prior to stage t_1. As $\hat{S}$ is infinite, y_0 and y_1 must exist. By (4.2), there is a stage $t_2 \geq t_1$ such that for all $t \geq t_2$ and $i \leq 1$, $\Psi_i^t(P_i^t; y_i)$ is defined, $P_i^t[\psi_i(y_i, t_2)] = P_i^{t_2}[\psi_i(y_i, t_2)]$, and $\ell_{\Delta_i, \Lambda_i, \Psi_i}(t) \geq y_i$.

Next, we fix $x_0, x_1 \in \hat{S}$ such that for $i = 0, 1$, $x_i > n, \omega(n, t_2), \psi_{1-i}(y_{1-i}, t_2)$, $x_i \notin D_i^{t_2+1}$, and x_0 and x_1 are greater than any numbers in any quadruple through which an attack on $\hat{R}$ was begun prior to stage t_2. As $\hat{S}$ is infinite, x_0 and x_1 must exist. By (4.3), there is a stage $t_3 \geq t_2$ such that for all $t \geq t_3$ and $i \leq 1$, $\Phi_i^t(V_{1-i}^t; x_i)$

is defined, $V^t_{1-i}[\phi_i(x_i, t_3)] = V^{t_3}_{1-i}[\phi_i(x_i, t_3)]$, $\ell_{\Gamma_i, \Xi_i, \Phi_i}(t) \geq x_i$, and (3.13) holds for $i \leq 1$.

We may assume that $\Theta^{t_3}(E^{t_3-1}; n)$ is undefined, else we are done. Under this assumption, all the conditions required for action for n in Step 1 of the construction hold at stage t_3; hence an attack on $\hat{R}$ for n will be begun at stage t_3, completing the proof of the induction step. □

By Lemma 4.7, by our assumption that $\hat{R}$ is not satisfied, and as $e \not\leq f$, we may fix the least n such that $\Theta(E; n) \downarrow \neq F(n)$. Note that as any newly declared axiom $\Theta^{s+1}(E^s; n) = k$ sets $k = F(n)$, this can only be the case if $n \in F$. We now look only at stages following the last stage s_0 at which a new axiom $\Theta^{s_0+1}(E^{s_0}; n)$ is declared. At this stage s_0, an attack on $\hat{R}$ for n through some $\langle x_0, x_1, y_0, y_1 \rangle$ is begun, and we fix the numbers in this quadruple. By choice of s_0, this attack will be cancelled only if it is replaced by another attack, so there is an uncancelled attack on $\hat{R}$ for n at all stages $t > s_0$.

In the next lemma, we will show that $\hat{R}$ is satisfied.

Lemma 4.8. *Fix the uncancelled attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1 \rangle$ which is the final replacement attack for the attack begun at stage s_0. Then there is a stage r at which we declare this attack to be successful. Thus $\hat{R}$ is satisfied.*

Proof. We first show that there is a stage r as specified in the lemma. We have chosen n so that at some stage $t \geq s_0$, $n \in F^t - F^{t-1}$. Thus by the remarks following (2.7), there must be an injury to $\Omega_k(P_k \oplus W_{1-k}; n)$ for $k = 0, 1$ at stage t. Now either there will be a j such that an attack on $\hat{R}$ for n through $\langle x_0, x_1, y_0, y_1 \rangle$ is begun before stage s_j, or we will begin such an attack at stage s_j. If this attack is completed before stage t, then it must have been declared to be successful, else it would have been cancelled and replaced by another attack. So assume that this attack is not completed before stage t. Now the attack will be declared to be successful if there is a first stage in $[s_j, t]$ at which there is an injury to $\Omega_{1-i}(P_{1-i} \oplus W_i; n)$; since t is such a stage, this must eventually happen. Thus there must be an r as specified in the lemma.

When an attack is begun at s_0, we have $\theta(n, s_0 + 1) \geq \phi_i(x_i, s_0)$ for $i = 0, 1$. As this attack is never cancelled, we have $E^s[\theta(n, s_0 + 1)] = E[\theta(n, s_0 + 1)]$ for all $s \geq s_0$, and so as $v_i \leq e$ for $i = 0, 1$,

$$(4.4) \quad V^s_{1-i}[\phi_i(x_i, s_0)] = V_{1-i}[\phi_i(x_i, s_0)] \text{ for all } s \geq s_0.$$

Now at the least stage $r > s_0$ at which an attack on $\hat{R}$ for n is declared to be successful, we place x_i into C^{r+1}_i for some $i \leq 1$, and note that $\Phi^{r-1}_i(V^{r-1}_{1-i}; x_i) \downarrow = 0$ for that i. By (4.3), there must be a stage $s \geq r$ at which there is an injury to $\Phi_i(V_{1-i}; x_i)$, contrary to (4.4). The lemma now follows. □

Theorem 1.2 follows immediately from Lemmas 4.3, 4.6, and 4.8 and the earlier comment that the theorem follows if all requirements are satisfied.

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