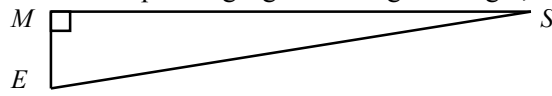


Hellenistic Astronomers and the Origins of Trigonometry: a brief look

Trigonometry was developed primarily as a tool for solving problems in Greek astronomy during the period 300 BCE–300 CE. The major figures in mathematical astronomy were Aristarchus of Samos (310–230 BCE), Hipparchus (180–125 BCE), Menelaus of Alexandria (circa 100 CE), and Claudius Ptolemy (85–165 CE). The trigonometric functions of today – sine, cosine, tangent, and their reciprocals – were defined and studied much later, but the seeds were sown by these scientists and some of their contemporaries.

To **Aristarchus**, the Sun was at the center of the solar system. (See Thomas L. Heath's book, *Aristarchus of Samos: the ancient Copernicus*.) About 260 BCE, Aristarchus devised a method for measuring the distance between the Earth and the Sun. He couldn't find an absolute distance but rather he found the ratio of the distances between the Earth and the Sun and the Earth and the Moon. Aristarchus observed that when the Moon is half full, the angle between the line of sight from Earth to Sun and the line of sight from Earth to Moon was a small amount less than a right angle. The corresponding figure is a right triangle,



where $\angle SEM$ is close to a right angle and hence $\angle ESM$ is a small angle. Aristarchus estimated $\angle ESM$ to be one-thirtieth of a quadrant (in our notation, $\left(\frac{90}{30}\right)^\circ = 3^\circ$). Since trigonometry had not yet been invented, Aristarchus couldn't then say that $ME/SE = \sin 3^\circ$ and then use a trig table or a calculator to find the numerical value of $\csc 3^\circ$ as his ratio. What he did do was to derive inequalities which he used to bound the ratio between two quantities. Aristarchus' inequalities are expressible in modern terms as saying that

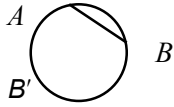
$$\text{If } \alpha \text{ and } \beta \text{ are acute angles and if } \beta < \alpha, \text{ then } \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}.$$

[These are impressive results. Observe that they say that on the open interval from 0 to a right angle, $(\sin x)/x$ is a decreasing function and $(\tan x)/x$ is an increasing function.] Aristarchus then showed that $1/20 < \sin 3^\circ < 1/18$; so the Sun is between 18 and 20 times as far from the Earth as the Moon is.

Today it is known that the Sun is in fact much farther from the Earth than that amount. The reason for the discrepancy is not in the mathematics — It's because $\angle ESM$ is smaller than the ancient estimate, close to $1/6^{\text{th}}$ of a degree. If one uses this value and Aristarchus' calculations, one gets a very good idea of the true value of the relation of the distances.

Clearly, for reasoning such as that of Aristarchus to be useful in solving other problems concerning the motion of the planets, comets, and stars, someone had to devise a method for finding values of the sine function and then do the calculations and make the results available. The major step was taken by **Hipparchus**. He did not use the sine or cosine functions, however. For Hipparchus and the later Greek writers, the fundamental trigonometric object was the chord of a circle as a function of the corresponding arc. Measuring the chord in units expressed in terms of the radius of the circle, Hipparchus made tables of the chord of an arc. In a twelve-volume book, he explained his methods of calculation, wrote out his table, and applied it to derive many facts about specific stars - including the locations of 850 'fixed stars', the length of the lunar month and the year, the size of the Moon, and other facts in astronomy. Unfortunately, nearly all of Hipparchus' writings are no longer in existence; what we know of them comes from much later commentaries about the work of Claudius Ptolemy, the later mathematical astronomer.

Menelaus studied geometry in a plane and on a sphere. (A theorem of advanced Euclidean geometry is named after him.) Along the way, he needed to deal with the relation of chord to arc.



When the radius OB is extended to a diameter BB' , the Pythagorean theorem can be applied to the triangle BAB' , since an angle inscribed in a semicircle is a right angle: $(AB)^2 + (AB')^2 = (\text{diameter})^2$. [Here AB is the chord of (arc AB) and AB' is the chord of (semicircle – arc AB). This is

essentially the fundamental identity of modern trigonometry, $\sin^2 \theta + \cos^2 \theta = 1$, since the cosine is the “complement’s sine”: $\sin(\pi / 2 - \theta) = \cos \theta$.] Both Menelaus and C. Ptolemy used this identity to derive many theorems.

Claudius Ptolemy wrote major books about geography and astronomy. In his cosmology, the Earth is at the center of the universe, and planetary orbits are slightly perturbed circular paths. Ptolemy continued the work of Hipparchus and used new trigonometric identities to make very detailed tables of values of the chord function. (These included a half-arc formula, like our $\sin(\theta / 2)$ formula. The half-arc formula contains a square root, so Ptolemy had to calculate numerous square roots, inevitably resulting in truncation or round-off errors of approximation.) These trigonometric tables were used in elaborate, often elegant, astronomical calculations.

Following Hipparchus (or perhaps Hipsicles, 180 BCE), he divided a circle into 360 equal parts (which his predecessors had called degrees). He used sexagesimal subdivisions of a part, to avoid, as he said, “the embarrassment of fractions.” (So the concept of a fraction apparently still required a splitting into a sum of distinct unit fractions.) For instance, he approximated $\text{crd } 36^\circ$ as $37^\text{p } 4' 55''$, with one chordal part being one sixtieth of a radius; since he usually referred to a radius of 60 units, this is $37;4,55$ units - in Neugebauer’s notation; otherwise, it is $0;37,4,55$ of a radius. He approximated the ratio later called pi by $3;8,30$ – which is $377/120$, or $3.141666\dots$

Ptolemy discovered spherical as well as planar trigonometric identities. Two of Ptolemy’s properties of chords of a circle correspond to the formulas for the sine of a sum and a difference of two angles. He applied these facts to prove what is now called Ptolemy’s Theorem for Cyclic Quadrilaterals, which says that given four successive points on a circle, the sum of the products of the lengths of the two sets of opposite sides equals the product of the diagonals.

All of the work mentioned above appeared in his book *Syntaxis Mathematica* [Mathematical Collection], some time around 150 CE. In his *A History of Mathematics*, Carl Boyer calls it “by far the most influential and significant trigonometric work of all antiquity.” The scientists who read the *Syntaxis* over the centuries considered it to be magnificent. Commentators called it magiste, or “greatest,” and after it was translated into Arabic, Moslem scholars called the collection *Al Magiste*. This linguistically mixed phrase meant “the greatest,” and gradually the title of Ptolemy’s masterpiece became the *Almagest*. This book remained the standard astronomy book for 1300 years.

Ptolemy wrote an eight volume *Geography*, in which he again applied geometry and trigonometry, as well as various methods for projecting the points on a sphere into a plane – including the method now known as ‘stereographic projection.’ Boyer remarks, “The importance of Ptolemy for geography can be gauged from the fact that the earliest maps in the Middle Ages that have come down to us in manuscripts, none before the thirteenth century, had as prototypes the maps made by Ptolemy more than a thousand years earlier.”

Let us finish this brief summary with a quotation from Claudius Ptolemy himself: “When I trace at my pleasure the windings to and fro of the heavenly bodies, I no longer touch the earth with my feet. I stand in the presence of Zeus himself and take my fill of ambrosia, food of the gods.”