ADVICE ON MATHEMATICAL WRITING

This handout lists some writing tips when you are preparing a math paper.

1. Follow good writing habits

A math paper should be punctuated, spelled, and organized clearly, just as in papers for disciplines besides math. One difference between a paper about history or literature and a paper about math is that math papers usually have shorter paragraphs. It is rare for a math paper to have a paragraph that takes up half a page. If you find a very long paragraph in your paper, see if there is more than one main idea in the paragraph (usually there will be) and break up the paragraph into new ones in a sensible way.

2. Notation

(1) Never begin a sentence with a symbol or a number; always start with a word. (Of course in *speaking* we may do this, but writing and speaking are not the same.)

Bad: Draw a line through P and Q. R is not on that line.
Good: Draw a line through P and Q. The point R is not on that line.

Bad: Let n be an even number. $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number. Thus $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number, so $n = 2m$ for some $m \in \mathbf{Z}$.

Bad: 2 is the smallest prime number.
Good: The smallest prime number is 2.

(2) If two symbols are not part of the same mathematical expression, they should never appear next to each other without words or punctuation marks between them.

Bad: If $n \neq 0$ $n^2 > 0$.
Good: If $n \neq 0$, $n^2 > 0$.
Good: If $n \neq 0$ then $n^2 > 0$.

(3) When introducing a new variable, use one which looks natural in that context (if possible). A lot of the time a choice is just common sense.

Bad: Let m be a prime.
Good: Let p be a prime.

Bad: Let X be a set, and pick an element of X, say t.
Good: Let X be a set, and pick an element of X, say x.

Bad: Pick two elements of the set X, say x and u.
Good: Pick two elements of the set X, say x and y.
Good: Pick two elements of the set X, say x_1 and x_2.
Good: Pick two elements of the set X, say x and x'.

(4) Always define new notation (is it a number? a function? of what type?) and be clear about its logical standing.

Very bad: Since n is composite, $n = ab$.
Bad: Since n is composite, $n = ab$ for some integers a and b.
Good: Since n is composite, $n = ab$ for some integers a and b greater than 1. [*Every* integer is a product, since $n = n \cdot 1$, so writing only "$n = ab$" because n is composite is saying nothing. Mentioning constraints on a and b as well, like $a > 1$ and $b > 1$, is giving some content to the equation.]

Bad: If a polynomial $f(x)$ satisfies $f(n) \in \mathbf{Z}$, does $f(x)$ have integer coefficients?
Good: If a polynomial $f(x)$ satisfies $f(n) \in \mathbf{Z}$ for every $n \in \mathbf{Z}$, does $f(x)$ have integer coefficients?

(5) Do not give multiple meanings to the same variable in a proof.

Bad: To show the sum of two even numbers is even, suppose a and b are even. Then $a = 2m$ and $b = 2m$, for some integer m. We have $a + b = 4m = 2(2m)$, which is even. [Notice this "proof" showed the sum of two even numbers is always a multiple of 4, which is nonsense.]
Good: To show the sum of two even numbers is even, suppose a and b are even. Then $a = 2m$ and $b = 2n$, for some integers m and n. We have $a + b = 2m + 2n = 2(m + n)$, which is even.

(6) Avoid overloading meaning into notation.

Bad: Let $x > 0 \in \mathbf{Z}$.
Good: Let x be an integer, with $x > 0$.
Good: Let x be a positive integer.

(7) **NEVER** use the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$ when writing, *except* in a paper on logic. Write out what you mean in ordinary language.

Bad: The conditions imply $a = 0 \wedge b = 1$.
Good: The conditions imply $a = 0$ and $b = 1$.

Bad: If $\exists$ a root of the polynomial then there is a linear factor.
Good: If there is a root of the polynomial then there is a linear factor.

Bad: If the functions agree at three points, they agree $\forall$ points.
Good: If the functions agree at three points, they agree at all points.

(8) Avoid silly abbreviations, or the misuse of standard notations. or the use of abbreviations which are used strictly on the blackboard (like WLOG, s.t., and iff).

Bad: When n is $\int$, $2n$ is an even number.
Good: When n is integral, $2n$ is an even number.
Good: When n is an integer, $2n$ is an even number.

Bad: Let z be a $\mathbf{C}$.
Good: Let z be a complex number.
Good: Choose $z \in \mathbf{C}$.

Bad: WLOG, we can assume $x > 0$.
Good: Without loss of generality, we can assume $x > 0$.

Bad: There is a point x s.t. $f(x) > 0$.

Good: There is a point x such that $f(x) > 0$.

(9) If a piece of notation is superfluous in your writing, don't use it.

Bad: Every differentiable function f is continuous.
Good: Every differentiable function is continuous.
Good: All differentiable functions are continuous.

Bad: A square matrix A is invertible when its determinant is not 0.
Good: A square matrix A is invertible when $\det A \neq 0$.
Good: A square matrix is invertible when its determinant is not 0.

The difference between the use of A in the Bad example and in the first Good example above is that in the first Good example something is actually done with A: we refer to it again in $\det A$. In the Bad example the use of A is superfluous notation.

3. Equations and expressions

An equation is anything with an equals sign: $3^2 + 4^2 = 5^2$ and $(x^2)' = 2x$ are equations. An expression, however, doesn't involve an equals sign. For example, $(a^3 + b^3)^2$ is an expression. Don't call $(a^3 + b^3)^2$ an equation: it is not saying two things are equal.

An important contrast between mathematical and ordinary text is the frequency with which equations or expressions are displayed on their own line. Compare the following two forms of the Pythagorean theorem:

- If the sides of a right triangle are a, b, and c, with c being the longest side, then $a^2 + b^2 = c^2$.
- If the sides of a right triangle are a, b, and c, with c being the longest side, then
$$a^2 + b^2 = c^2.$$

Both of these are fine, but writing the conclusion of the Pythagorean theorem as a centered equation on its own line, in the second version, brings out more visibly its significance. Not all equations have to be on their own lines, of course, but this style pervades mathematical (and other scientific) writing. Anyone who has never written a mathematics paper before needs to have the idea of displaying mathematics on its own line firmly in mind and use it often.

As another example, compare the way a parametric formula is described:

- The general formula for a primitive Pythagorean triple is $(m^2 - n^2, 2mn, m^2 + n^2)$, where m and n are integers with $m > n$.
- The general formula for a primitive Pythagorean triple is
$$(m^2 - n^2, 2mn, m^2 + n^2),$$
where m and n are integers with $m > n$.

In the second version, placing the general formula on its own line makes it easier for the reader to absorb the information. Notice the sentence continues after the displayed formula, on a new line.

Equations that will be referred to later in a paper should have an equation number placed to the left, and that will serve as the reference label. For example, we could write

$$(3.1) \qquad (x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

and when this is cited later, it can be called equation (3.1). Equations that are not referred to later do not need to be labelled in the margin.

Here are two comments on the use of displayed equations and expressions.

(1) When presenting the steps of a computation, where there are several steps, usually they are displayed on successive lines and the left side is not repeated since it does not change.

Bad: Next we see $(x+1)^3 = (x+1)^2(x+1) = (x^2+2x+1)(x+1) = x^3+3x^2+3x+1$.
Bad: Next we see

$$\begin{aligned} (x+1)^3 &= (x+1)^2(x+1) \\ (x+1)^3 &= (x^2+2x+1)(x+1) \\ (x+1)^3 &= x^3+3x^2+3x+1. \end{aligned}$$

Good: Next we see

$$\begin{aligned} (x+1)^3 &= (x+1)^2(x+1) \\ &= (x^2+2x+1)(x+1) \\ &= x^3+3x^2+3x+1. \end{aligned}$$

(2) Equations and expressions do not stand by themselves. They appear as part of a sentence and should be punctuated accordingly! If an equation ends a sentence, place a period at the end of the line. If an equation appears in the middle of a sentence, use a comma after the equation if one would naturally pause there. Sometimes no punctuation is needed after the equation.

Bad: The general formula for a primitive Pythagorean triple is

$$(m^2-n^2, 2mn, m^2+n^2)$$

where m and n are integers with $m > n$.
Good: The general formula for a primitive Pythagorean triple is

$$(m^2-n^2, 2mn, m^2+n^2),$$

where m and n are integers with $m > n$.

Bad: We call x_0 a *critical point* of f when f is differentiable and

$$f'(x_0) = 0$$

.

Good: We call x_0 a *critical point* of f when f is differentiable and

$$f'(x_0) = 0.$$

Good: When f is differentiable, and x_0 satisfies

$$f'(x_0) = 0,$$

we call x_0 a *critical point*.
Good: When f is differentiable, any x_0 where

$$f'(x_0) = 0$$

is called a *critical point*.

4. Parentheses

(1) Avoid pointless parentheses in mathematical expressions.

Bad: $(x+y)(x-y) = (x^2 - y^2)$. [The parentheses on the right have no purpose.]
Good: $(x+y)(x-y) = x^2 - y^2$.

Bad: If 7 is a factor of the product $(a_1 a_2 \cdots a_n)$, then ...
Good: If 7 is a factor of the product $a_1 a_2 \cdots a_n$, then ...

Bad: The length is a factor of $(p-1)$.
Good: The length is a factor of $p-1$.

Bad: The Taylor series for $\log(1+x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n}.$$

Good: The Taylor series for $\log(1+x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Good: $(a+b)^2 - (a+c)^2 = b^2 - c^2 + 2ab - 2ac$.
Good: $(a+b)^2 - (a+c)^2 = (b^2 - c^2) + 2ab - 2ac$. [This example is good *only* if the writer wants the reader to view $b^2 - c^2$ as a single part of the right side.]

(2) Use parentheses to avoid confusing the meaning between a subtraction sign and a negative sign in a mathematical expression.

Very bad: $(a+b) - c = -ac - bc$. [If you look at the right side, you can see the writer meant for the left side to be the product of $a+b$ and $-c$, but the left side instead looks like "a plus b minus c."
Bad: $(a+b) \cdot -c = -ac - bc$.
Good: $(a+b)(-c) = -ac - bc$.

5. Commas

(1) Commas are natural places to pause briefly, but not as fully as a period. If you read something in your head, you should be able to notice badly placed commas, either because no pause should occur or because a period should be there instead of a comma.

Bad: The condition we want is, $a = 2b$.
Good: The condition we want is $a = 2b$.

Bad: The set is infinite, we pick a large finite subset of it.
Good: The set is infinite. We pick a large finite subset of it.

(2) While "If ..., then..." is a common phrase, it is bad English to write "Let..., then...."

Bad: Let n be an even number, then $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number. Then $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number, so $n = 2m$ for some $m \in \mathbf{Z}$.

(3) Be careful about starting sentences with "Which". Usually they should be part of the preceding sentence.

Bad: The sequence $\{x_n\}$ tends to 1 as $n \to \infty$. Which means $x_n > 0$ for large n.
Good: The sequence $\{x_n\}$ tends to 1 as $n \to \infty$, which means $x_n > 0$ for large n.

Bad: Let k be an integer. Which lies between 10 and 100.
Good: Let k be an integer which lies between 10 and 100.

6. Use helpful words

(1) Tell the reader where you are going.

Good: We will prove this by induction on n.
Good: We will prove this by induction on the dimension.
Good: We argue by contradiction.
Good: Now we consider the converse direction.
Good: But $f(x)$ is actually continuous. To see why, consider...
Good: The inequality $a \leq b$ is strict: $a < b$. Indeed, if there was equality then...

(2) Use key words to show the reader how you are reasoning. These include

since, because, on the other hand, observe, note.

At the same time, vary your choice of words to avoid monotonous writing. This may require you to completely rewrite a paragraph.

Bad: We proved, for any a, that if a^2 is even, then a is even. Now suppose a^8 is even. Since $a^8 = (a^4)^2$, we obtain that a^4 is even. Then a^2 is even. Then a is even.

Good: We proved, for any a, that if a^2 is even, then a is even. Now suppose a^8 is even. Then, by successively applying the result we proved to a^4, a^2, and a, we see that a is even.

(3) Watch your spelling! If you aren't sure of the difference between "necessary" and "neccessary" or "discriminate" and "discriminant," look it up. (Canadian students may use their own flavour of spelling, but non-native English speakers should be careful not to let the grammatical rules of their native language affect their writing in English where those rules are different.)

Use "it's" only to mean "it is". The word "its", like "his" and "her", refers to possession.

Bad: It's clear that $f(x)$ has a real root since it's degree is odd.
Bad: Its clear that $f(x)$ has a real root since its degree is odd.
Good: It's clear that $f(x)$ has a real root since its degree is odd.
Good: Since $f(x)$ has odd degree, clearly it has a real root. [Write like this if you can't remember the difference between its and it's.]

Bad: Its surely true that starting your final draft on the last day will leave its mark in your work.

Good: It's surely true that starting your final draft on the last day will leave its mark in your work.

7. Fonts

Here are a few points about italic and non-italic fonts in mathematical writing.

(1) Single letters in math (used as variables, parameters, or function names) are always typed in italics:

Bad: The roots of $\mathrm{ax}^2 + \mathrm{bx} + \mathrm{c}$ are

$$\frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}.$$

Good: The roots of $ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Bad: The complex number $\mathrm{a}+\mathrm{bi}$ has positive imaginary part.
Good: The complex number $a + bi$ has positive imaginary part

Bad: One solution of the differential equation $\mathrm{f'(x)} = \mathrm{f(x)}$ is e^{x}.
Good: One solution of the differential equation $f'(x) = f(x)$ is e^x.

Bad: The nth Fibonacci number is F_{n}.
Bad: The nth Fibonacci number is F_n.
Good: The nth Fibonacci number is F_n.

(2) Numbers are never typed in italics. Italic numbers look awful and should be avoided in all circumstances.

Bad: Now we will differentiate $x^2 - \mathit{3}x + \mathit{1}$.
Good: Now we will differentiate $x^2 - 3x + 1$.

(3) Traditional function names that involve several letters are not typed in italics.

Bad: The basic trigonometric functions are $\mathit{sin}\ \theta$ and $\mathit{cos}\ \theta$.
Good: The basic trigonometric functions are $\sin\theta$ and $\cos\theta$.

Bad: For positive a and b, $\mathit{log}\ ab = \mathit{log}\ a + \mathit{log}\ b$.
Good: For positive a and b, $\log ab = \log a + \log b$.

(4) A new term being defined for the first time is put in italics. (Some books put defined terms in **bold** rather than italics.)

Bad: The dimension of a vector space is the size of any basis.
Good: The *dimension* of a vector space is the size of any basis.

Bad: The set which contains no elements is called the empty set.
Good: The set which contains no elements is called the *empty set*.

8. Types of Mathematical Results

In mathematics, results are labelled as either a theorem, lemma, or corollary. What's the difference?

- A theorem is a main result.
- A lemma is a result whose primary purpose is to be used in the proof of a theorem but which, on its own, is not considered significant or as interesting.
- A corollary is a result that follows from a theorem. It could be a special case of the theorem or a particularly important consequence of it.

So theorems stand on their own, a lemma always comes before a theorem, and corollaries always come after a theorem. The order in which these appear, then, is always

Lemma, Theorem, Corollary.

There is no reason a theorem must have a lemma before it or a corollary after it. But if you have a string of lemmas which don't lead to a theorem, for instance, then it will look strange to anyone experienced with mathematical writing.

Here are two examples. First we give a lemma and a theorem whose proof depends on the lemma.

Lemma 8.1. *In the integers, if d is a factor of a and b then d is a factor of $ax + by$ for any integers x and y.*

Proof. Since d is a factor of both a and b, we can write $a = dm$ and $b = dn$ for some integers m and n. Then for any x and y we have

$$ax + by = dmx + dny = d(mx + ny),$$

which shows d is a factor of $ax + by$. □

Theorem 8.2. *If a and b are integers and $ax_0 + by_0 = 1$ for some integers x_0 and y_0, then a and b have no common factor greater than 1.*

Proof. This will be a proof by contradiction. Suppose there is a common factor $d > 1$ of a and b. Applying Lemma 8.1 to the particular combination $ax_0 + by_0$, d is a factor of $ax_0 + by_0$, so d is a factor of 1. But there are no factors of 1 which are greater than 1, so we have a contradiction. Therefore a and b have no common factor greater than 1. □

Therem 8.2 uses Lemma 8.1, but the statement of Lemma 8.1 was deemed (by the author writing it) to be worth isolating on its own. So it becomes a lemma rather than appear completely inside the proof of Theorem 8.2. Perhaps the author anticipates other uses of Lemma 8.1, and so wants to state it separately.

Next we give a theorem in linear algebra and a corollary which follows from the theorem.

Theorem 8.3. *For any two square matrices A and B, $\det(AB) = (\det A)(\det B)$.*

[The proof of this is hard and is not included here.]

Corollary 8.4. *An invertible matrix has a nonzero determinant.*

Proof. If A is invertible, say of size $n \times n$, then $AB = I_n$ for some matrix B. Taking the determinant of both sides, Theorem 8.3 tells us

$$(\det A)(\det B) = \det(I_n) = 1,$$

so $\det A \neq 0$. □

Why do we need lemmas at all? Could we call everything a theorem? Yes, but the point of the three different names (lemma, theorem, corollary) is to indicate to the reader how the writer views the comparative standing of the different results.

Although lemmas are principally intended to be used for the proof of a more important result, sometimes a lemma turns out to be a very significant result and is even named after someone, but is still referred to as a lemma for historical reasons. Examples include Hensel's lemma (in number theory), Nakayama's lemma (in commutative algebra), the Riemann–Lebesgue lemma (in harmonic analysis), and the Schwarz lemma (in complex analysis).

In addition to the mathematical results in a paper, terminology is used and may need to be defined for the reader. A definition is not a theorem: you don't prove a definition. Here are two definitions.

Definition 8.5. A *geodesic* is a curve that locally minimizes lengths between points.

Definition 8.6. When all the elements in a partially ordered set are comparable to each other, we call it a *totally ordered* set.

Here is a bad definition.

Definition 8.7. For differentiable functions $f(x)$ and $g(x)$, the derivative of their sum $f(x) + g(x)$ is $f'(x) + g'(x)$.

Why isn't this a definition? Because the derivative is defined using limits, and you need to *prove* that $f'(x)+g'(x)$ is really the value of the derivative of $f(x)+g(x)$. In other words, this so-called definition is in fact a theorem. Don't label a theorem as a definition.

9. Latin Abbreviations

There are several abbreviations from Latin which are confused with each other if you don't know what they mean. Here they are.

- *i.e.* means "that is" (id est),
- *e.g.* means "for example" (exempli gratia),
- *etc.* means "and other *things*" (et cetera),
- *et al.* means "and other *people*" (et alii).

Use *i.e.* to make something more specific and *e.g.* to give an example (special cases of something more general). Some people write *i.e.* by mistake instead of *e.g.* to begin an example or *etc.* instead of *et al.* by mistake in a list of people.

It is standard American English usage to place a comma after these abbreviations.

Bad: The equation $a^2 + b^2 = c^2$ has infinitely many solutions in positive integers, *i.e.*, $(3, 4, 5)$ and $(5, 12, 13)$.

Bad: The equation $a^2 + b^2 = c^2$ has infinitely many solutions in positive integers, *e.g.*, $(3, 4, 5)$ and $(5, 12, 13)$.

Bad: The equation $y^2 - y = x^3$ has only trivial integral solutions, *e.g.*, $(x, y) = (0, 0)$ and $(0, 1)$.

Good: The equation $y^2 - y = x^3$ has only trivial integral solutions, *i.e.*, $(x, y) = (0, 0)$ and $(0, 1)$.

Bad: Binary quadratic forms were studied by Gauss, Lagrange, Minkowski, *etc.*

Good: Binary quadratic forms were studied by Gauss, Lagrange, Minkowski, *et al.*

10. Proofread

Proofread your paper before you submit it (whether as a draft or in its final version). Put yourself in the frame of mind of your desired target audience, to whom the material in the paper will not be as familiar as it is to you. What you say should be understandable to the target audience. Points to keep in mind are:

(1) Do the ideas flow along in a logical manner?
(2) Is there proper motivation for new ideas?
(3) Are there suitable examples or counterexamples?

(4) Does the math make sense? You certainly don't want logical or computational errors. And mathematical reasoning should be complete: don't end a proof halfway through without an explanation.
(5) Is it written correctly? Check for grammatical mistakes, misspelled words (a spell-checker won't notice that "too" should be "two" or "of" should be "or"), wrong punctuation, incorrect references (Theorem 2.2 when you means Theorem 2.1), and inconsistent notation (a_n on one line shouldn't become x_n on the next line).