ORDERS OF GROWTH

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Date: Sept. 2, 2009.

1. Introduction

Gaining an intuitive feel for the relative growth of functions is important if you really want to understand their behavior. It also helps you better grasp topics in calculus such as convergence of improper integrals and infinite series.

We want to compare the growth of three different kinds of functions of x, as $x \to \infty$:

- power functions x^r for $r > 0$ (such as x^3 or $\sqrt{x} = x^{1/2}$),
- exponential functions a^x for $a > 1$,
- logarithmic functions $\log_b x$ for $b > 1$.

Some examples are plotted in Figure 1 over the interval $[1, 10]$. The relative sizes are quite different for x near 1 and for larger x. (Some coefficients are included on 2^x and x^2 to keep them from blowing up too quickly in the picture.)

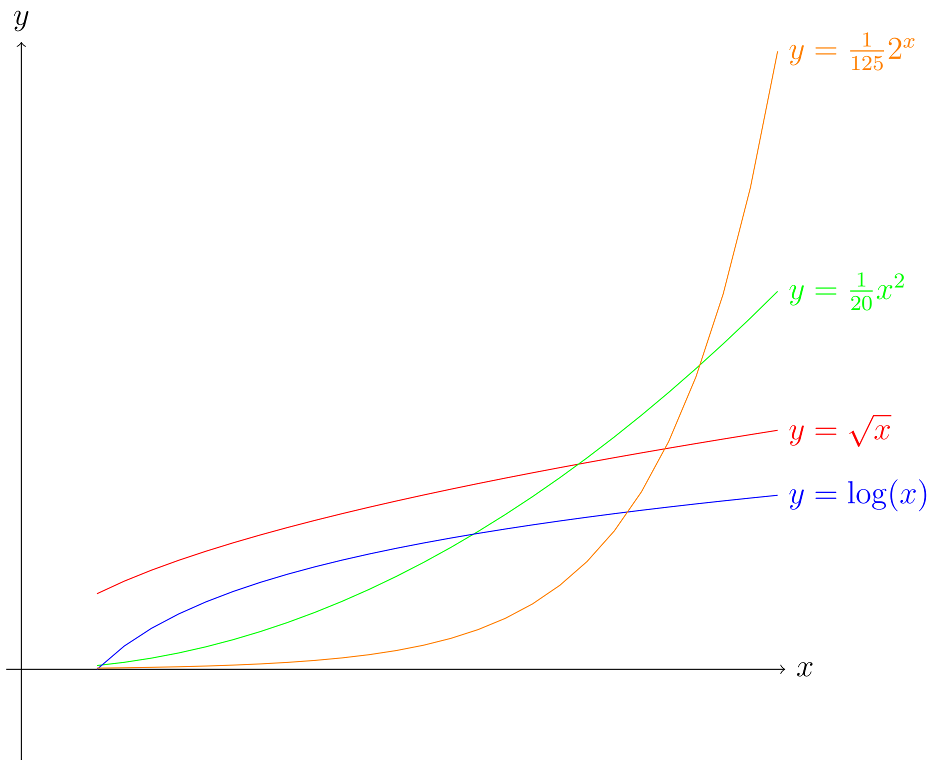

Figure 1. Graphs of several functions for $x \in [1, 10]$

All power functions, exponential functions, and logarithmic functions (as defined above) tend to ∞ as $x \to \infty$. But these three classes of functions tend to ∞ at *different* rates. The main result we want to focus on is the following one. It says e^x grows faster than *any* power function while $\log x$ grows slower than *any* power function. (A power function means x^r with $r > 0$, so $1/x^2 = x^{-2}$ doesn't count.)

Theorem 1.1. *For each* $r > 0$, $\displaystyle\lim_{x\to\infty} \frac{x^r}{e^x} = 0$ *and* $\displaystyle\lim_{x\to\infty} \frac{\log x}{x^r} = 0$.

This is illustrated in Figure 2. At first the functions are increasing, but for larger x they tend to 0.

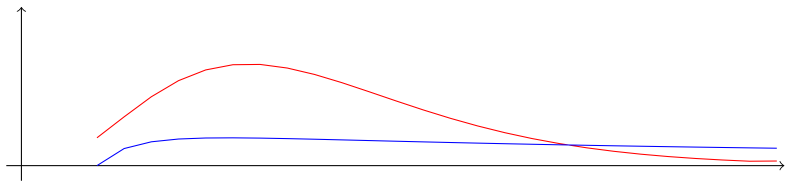

FIGURE 2. Graphs of x^3/e^x and $\log(x)/x$ for $x \in [1, 10]$

After we prove Theorem 1.1 and look at some consequences of it in Section 2, we will compare power, exponential, and log functions with the sequences $n!$ and n^n and eventually show that between any two functions with different orders of growth we can insert infinitely many functions with different orders of growth between them.

2. Proof of Theorem 1.1 and some corollaries

Proof. (of Theorem 1.1) First we focus on the limit $x^r/e^x \to 0$. When $r = 1$ this says

$$\frac{x}{e^x} \to 0 \text{ as } x \to \infty. \tag{2.1}$$

This result follows from L'Hopital's rule.

To derive the general case from this special case, write

$$\frac{x^r}{e^x} = r^r \left(\frac{x/r}{e^{x/r}}\right)^r. \tag{2.2}$$

With r staying fixed, as $x \to \infty$ also $x/r \to \infty$, so $(x/r)/e^{x/r} \to 0$ by (2.1) with x/r in place of x. Then the right side of (2.2) tends to 0 as $x \to \infty$, so we're done.

Now we show $(\log x)/x^r \to 0$ as $x \to \infty$. Writing y for $\log(x^r) = r \log x$,

$$\frac{\log x}{x^r} = \frac{y/r}{e^y} = \frac{1}{r} \cdot \frac{y}{e^y}.$$

As $x \to \infty$, also $y \to \infty$, so $(1/r)(y/e^y) \to 0$ by (2.1) with y in place of x. Thus $(\log x)/x^r \to 0$. $\square$

Corollary 2.1. *For any polynomial $p(x)$, $\displaystyle\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$.*

Proof. By Theorem 1.1, for any $k > 0$ we have $x^k/e^x \to 0$ as $x \to \infty$. This is also true when $k = 0$. Writing $p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, we have

$$\frac{p(x)}{e^x} = a_d \frac{x^d}{e^x} + a_{d-1}\frac{x^{d-1}}{e^x} + \cdots + a_1 \frac{x}{e^x} + a_0 \frac{1}{e^x}.$$

Each x^k/e^x appearing here tends to 0 as $x \to \infty$, so $p(x)/e^x$ tends to 0 as $x \to \infty$. $\square$

Corollary 2.2. *For any $r > 0$ and $k > 0$, $\displaystyle\lim_{x\to\infty} \frac{(\log x)^k}{x^r} = 0$.*

Proof. Let $y = \log(x^r) = r \log x$, so

$$\frac{(\log x)^k}{x^r} = \frac{y^k/r^k}{e^y} = \frac{1}{r^k} \cdot \frac{y^k}{e^y}.$$

As $x \to \infty$, also $y \to \infty$. Therefore $(1/r^k)(y^k/e^y) \to 0$ by Theorem 1.1 (since $r^k > 0$). $\square$

We derived Corollary 2.2 from Theorem 1.1, but the argument can be reversed too. (So we could consider Corollary 2.2 as the main result and Theorem 1.1 as a consequence of it.) Take $k = 1$ in Corollary 2.2 to get the log part of Theorem 1.1 and use the change of variables $y = e^x$ in x^r/e^x to get the exponential part of Theorem 1.1 from Corollary 2.2. Specifically, when $y = e^x$

$$\frac{x^r}{e^x} = \frac{(\log y)^r}{y},$$

and as $x \to \infty$ we have $y = e^x \to \infty$, so by Corollary 2.2 we get $(\log y)^r/y \to 0$. Therefore $x^r/e^x \to 0$ as $x \to \infty$.

Corollary 2.3. *For any nonconstant polynomial $p(x)$ and positive k,* $\lim_{x\to\infty} \frac{(\log x)^k}{p(x)} = 0.$

Proof. For large x, $p(x) \neq 0$ since nonzero polynomials have only a finite number of roots. Write $p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, where $d > 0$ and $a_d \neq 0$. Then

$$\frac{(\log x)^k}{p(x)} = \frac{(\log x)^k}{x^d} \cdot \frac{1}{a_d + a_{d-1}/x + \cdots + a_0/x^d}.$$

As $x \to \infty$, the first factor tends to 0 by Corollary 2.2 while the second factor tends to $1/a_d \neq 0$, so the product tends to 0. $\square$

Corollary 2.4. *As $x \to \infty$, $x^{1/x} \to 1$.*

Proof. The logarithm of $x^{1/x}$ is $\log(x^{1/x}) = (\log x)/x$, which tends to 0 as $x \to \infty$. Exponentiating,

$$x^{1/x} = e^{(\log x)/x} \to e^0 = 1.$$

$\square$

Replacing e^x with a^x for any $a > 1$ and $\log x$ with $\log_b x$ for any $b > 1$ leads to completely analogous results.

Theorem 2.5. *Fix real numbers $a > 1$ and $b > 1$. For any $r > 0$ and integer $k > 0$,*

$$\lim_{x\to\infty} \frac{x^r}{a^x} = 0, \quad \lim_{x\to\infty} \frac{(\log_b x)^k}{x^r} = 0.$$

Proof. To deduce this theorem from the earlier results, write $a^x = e^{(\log a)x}$ and $\log_b x = (\log x)/(\log b)$. The numbers $\log a$ and $\log b$ are *positive.* Then, for instance, if we set $y = (\log a)x$,

$$\frac{x^r}{a^x} = \frac{x^r}{e^{(\log a)x}} = \frac{1}{(\log a)^r}\frac{y^r}{e^y}.$$

When $x \to \infty$, also $y \to \infty$ since $\log a > 0$, so the behavior of x^r/a^x follows from that of y^r/e^y using Theorem 1.1. Since $\log_b x = (\log x)/(\log b)$ is a constant multiple of $\log x$, carrying over the results on $\log x$ to $\log_b x$ is just a matter of rescaling. For instance, if we set $y = \log x$, so $\log_b x = y/\log b$, then

$$\frac{(\log_b x)^k}{x^r} = \frac{y^k/(\log b)^k}{e^{ry}} = \frac{1}{(\log b)^k}\frac{y^k}{(e^r)^y}.$$

As $x \to \infty$, also $y \to \infty$, so the exponential function $(e^r)^y$ dominates over the power function y^k: $y^k/(e^r)^y \to 0$. Therefore $(\log_b x)^k/x^r \to 0$ as $x \to \infty$. $\square$

For any nonconstant polynomial $p(x)$, it follows from Theorem 2.5 that

$$\lim_{x\to\infty}\frac{p(x)}{a^x}=0,\quad \lim_{x\to\infty}\frac{(\log_b x)^k}{p(x)}=0$$

in the same way we proved Corollaries 2.1 and 2.3.

3. Growth of basic sequences

We want to compare the growth of five kinds of sequences:

- power sequences n^r for $r>0$: $1, 2^r, 3^r, 4^r, 5^r, \dots$
- exponential sequences a^n for $a>1$: $a, a^2, a^3, a^4, a^5, \dots$
- log sequences $\log_b n$ for $b>1$: $0, \log_b 2, \log_b 3, \log_b 4, \log_b 5, \dots$
- $n!$: $1, 2, 6, 24, 120, \dots$
- n^n: $1, 4, 27, 256, 3125, \dots$

The first three sequences are just the functions we have already treated, except the real variable x has been replaced by an integer variable n. That is, we are looking at those old functions at integer values of x now.

Some notation to convey dominanting rates of growth will be convenient. For two sequences x_n and y_n, write $x_n \prec y_n$ to mean $x_n/y_n \to 0$ as $n\to\infty$. In other words, x_n grows *substantially slower* than y_n (if it just grew at half the rate, for instance, then x_n/y_n would be around $1/2$ rather than tend to 0). For instance, $n \prec n^2$ and $\sqrt{n} \prec n$. (The notation $\prec$ is taken from [1, Chap. 9], which has a whole chapter on orders of growth.)

Remark 3.1. The notation $x_n \prec y_n$ does *not* mean $x_n < y_n$ for all n. Maybe some initial terms in the x_n sequence are larger than the corresponding ones in the y_n sequence, but this will eventually stop and the *long term growth* of y_n dominates. For instance, $1000000n \prec n^2$ even though $n^2 < 1000000n$ for all small n. Indeed, the ratio

$$\frac{1000000n}{n^2}=\frac{1000000}{n}$$

tends to 0 as $n\to\infty$, but the ratio is not small until n gets quite large.

Theorem 1.1 tells us that

$$\log n \prec n^r \prec e^n \tag{3.1}$$

for any $r>0$. By Theorem 2.5, we can say more generally that

$$\log_b n \prec n^r \prec a^n \tag{3.2}$$

for any $a>1$ and $b>1$. How do the sequences $n!$ and n^n fit into (3.2)? They belong on the right, as follows.

Theorem 3.2. *For any $a>1$, $a^n \prec n! \prec n^n$. Equivalently,*

$$\lim_{n\to\infty}\frac{a^n}{n!}=0,\quad \lim_{n\to\infty}\frac{n!}{n^n}=0.$$

Proof. To compare a^n and $n!$, we use Euler's integral formula for $n!$:

$$n! = \int_0^\infty x^n e^{-x}\,\mathrm{d}x.$$

This integral has for a *lower* bound the same integral carried out just over $[0, n]$:

$$n! > \int_0^n x^n e^{-x} \mathrm{d}x.$$

On the interval $[0, n]$, e^{-x} has its smallest value at the right end: $e^{-x} \geq e^{-n}$. Therefore $x^n e^{-x} \geq x^n e^{-n}$ on $[0, n]$. Integrating both sides of this inequality from $x = 0$ to $x = n$ gives

$$\begin{aligned} \int_0^n x^n e^{-x} \mathrm{d}x &\geq \int_0^n x^n e^{-n} \mathrm{d}x \\ &= \frac{1}{e^n} \int_0^n x^n \mathrm{d}x \\ &= \frac{1}{e^n} \frac{n^{n+1}}{n+1} \\ &= \left(\frac{n}{e}\right)^n \frac{n}{n+1}. \end{aligned}$$

Therefore $n! > \left(\frac{n}{e}\right)^n \frac{n}{n+1}$, so

$$\frac{a^n}{n!} < \frac{a^n}{(n/e)^n (n/(n+1))} = \left(\frac{ae}{n}\right)^n \frac{n+1}{n}.$$

This final expression is an upper bound on $a^n/n!$. How does it behave as $n \to \infty$? For large n, $ae/n \leq 1/2$, so $(ae/n)^n \leq (1/2)^n$. Therefore $(ae/n)^n \to 0$. Since the other factor $(n+1)/n$ tends to 1, we see our upper bound on $a^n/n!$ tends to 0, so $a^n/n! \to 0$ as $n \to \infty$.

To show the other part of the theorem, that $n!/n^n \to 0$ as $n \to \infty$, we will get an *upper* bound on $n!$ and divide the upper bound by n^n. Write e^{-x} as $e^{-x/2}e^{-x/2}$ in Euler's factorial integral:

$$n! = \int_0^\infty x^n e^{-x} \mathrm{d}x = \int_0^\infty (x^n e^{-x/2}) e^{-x/2} \mathrm{d}x.$$

The function $x^n e^{-x/2}$ drops off to 0 as $x \to \infty$. Where does it have its maximum value? The derivative is $x^{n-1} e^{-x/2}(n - x/2)$ (check this), so $x^n e^{-x/2}$ vanishes at $x = 2n$. Checking the signs of the derivative to the left and right of $x = 2n$, we see $x^n e^{-x/2}$ has a maximum value at $x = 2n$, where the value is $(2n)^n e^{-n}$. Therefore $x^n e^{-x/2} \leq (2n)^n e^{-n}$ for all $x > 0$, so

$$\begin{aligned} n! &= \int_0^\infty (x^n e^{-x/2}) e^{-x/2} \mathrm{d}x \\ &\leq \int_0^\infty (2n)^n e^{-n} e^{-x/2} \mathrm{d}x \\ &= (2n)^n e^{-n} \int_0^\infty e^{-x/2} \mathrm{d}x \\ &= (2n)^n e^{-n} \cdot 2. \end{aligned}$$

Dividing throughout by n^n gives

$$\frac{n!}{n^n} \leq 2 \left(\frac{2}{e}\right)^n.$$

Since $2 < e$, the right side tends to 0, so $n!/n^n \to 0$ as $n \to \infty$. $\square$

Remark 3.3. In the proof we showed
$$\frac{n}{n+1}\left(\frac{n}{e}\right)^n \leq n! \leq 2^{n+1}\left(\frac{n}{e}\right)^n.$$
The true order of magnitude of $n!$ is $(n/e)^n\sqrt{2\pi n}$, by Stirling's formula [2, pp. 116–123].

The fact that $a^n \prec n! \prec n^n$ is intuitively reasonable, for the following reason: each of these expressions ($a^n, n!$, and n^n) is a product of n numbers, but the nature of these numbers is different. In a^n, all n numbers are the same value a, which is independent of n:
$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}.$$
In $n!$, the n numbers are the integers from 1 to n:
$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$
Since the terms in this product keep growing, while the terms in a^n stay the same, it makes sense that $n!$ grows faster than a^n (at least once n gets larger than a). In n^n, all n numbers equal n:
$$n^n = \underbrace{n \cdot n \cdots n}_{n \text{ times}}.$$
Since all the terms in this product equal n, while in $n!$ the terms are the numbers from 1 to n, it is plausible that n^n grows a lot faster than $n!$.

To summarize our results on sequences, we combine (3.2) and Theorem 3.2:
$$\log_b n \prec n^r \prec a^n \prec n! \prec n^n$$
Here $a > 1$, $b > 1$, and $r > 0$ (not just $r > 1$!). All sequences here tend to ∞ as $n \to \infty$, but the rates of growth are all different: any sequence which comes to the left of another sequence on this list grows at a substantially smaller rate, in the sense that the ratio tends to 0.

For example, can we find a (natural) sequence whose growth is intermediate between n and n^r for *every* $r > 1$? That is, we want to find a single sequence of numbers x_n such that $n \prec x_n \prec n^r$ for every $r > 1$. One choice is $x_n = n \log n$. Indeed,
$$\frac{n}{n \log n} = \frac{1}{\log n} \to 0,$$
so $n \prec n \log n$, and for any $r > 1$
$$\frac{n \log n}{n^r} = \frac{\log n}{n^{r-1}},$$
which tends to 0 since $r - 1 > 0$ and $\log n$ grows slower than *any* power function (with a positive exponent) by Theorem 1.1.

Using powers of $\log n$, we can write down infinitely many sequences with different rates of growth between n and every sequence n^r for $r > 1$:
$$n \prec n \log n \prec n(\log n)^2 \prec n(\log n)^3 \prec \cdots \prec n(\log n)^k \prec \cdots \prec n^r,$$
where k runs through the positive integers.

Is it possible to insert infinitely many sequences with different rates of growth between any two sequences with different rates of growth?

Theorem 3.4. *If $x_n \prec y_n$, there are sequences $\{z_n^{(1)}\}, \{z_n^{(2)}\}, \{z_n^{(3)}\}, \ldots$ such that*

$$x_n \prec z_n^{(1)} \prec z_n^{(2)} \prec z_n^{(3)} \prec \cdots \prec y_n.$$

Proof. Since $x_n/y_n \to 0$, for large n the ratio x_n/y_n is small. Specifically, $0 < x_n/y_n < 1$ for large n. For small positive numbers, taking roots makes them larger but less than 1:

$$0 < a < 1 \Longrightarrow 0 < a < \sqrt{a} < \sqrt[3]{a} < \cdots < \sqrt[k]{a} < \cdots < 1.$$

Since $x_n/y_n < 1$ for large n, this presents us with the inequalities

$$0 < \frac{x_n}{y_n} < \sqrt{\frac{x_n}{y_n}} < \sqrt[3]{\frac{x_n}{y_n}} < \cdots < \sqrt[k]{\frac{x_n}{y_n}} < \cdots < 1$$

for large n and $k = 1, 2, 3, \ldots$. Multiply through by y_n:

$$0 < x_n < \sqrt{x_n}\sqrt{y_n} < x_n^{1/3} y_n^{2/3} < \cdots < x_n^{1/k} y_n^{1-1/k} < \cdots < y_n. \tag{3.3}$$

For $k < \ell$, the ratio of the k-th root sequence to the ℓ-th root sequence is

$$\frac{x_n^{1/k} y_n^{1-1/k}}{x_n^{1/\ell} y_n^{1-1/\ell}} = \left(\frac{x_n}{y_n}\right)^{1/k-1/\ell}.$$

Since $1/k - 1/\ell > 0$, this ratio tends to 0 as $n \to \infty$. Therefore (3.3) leads to infinitely many sequences with growth intermediate between $\{x_n\}$ and $\{y_n\}$, namely the sequences $z_n^{(k)} = x_n^{1/k} y_n^{1-1/k}$ for $k = 2, 3, 4, \ldots$:

$$x_n \prec \sqrt{x_n}\sqrt{y_n} \prec x_n^{1/3} y_n^{2/3} \prec \cdots \prec x_n^{1/k} y_n^{1-1/k} \prec \cdots \prec y_n. \tag{3.4}$$

(If you want to label the first sequence with $k = 1$, set $z_n^{(k)} = x_n^{1/(k+1)} y_n^{1-1/(k+1)}$ for $k = 1, 2, 3, \ldots$.) □

The difference between (3.3) and (3.4) is that (3.3) is a set of inequalities which is valid for large n (namely n large enough to have $x_n/y_n < 1$), while (3.4) is a statement about rates of growth between different sequences: it makes no sense to ask if (3.4) is true at a particular value of n, any more than it would make sense to ask if the limit relation $\frac{n}{n+1} \to 1$ is "true" at $n = 45$.

References

[1] D. E. Knuth, R. Graham, O. Patashnik, "Concrete Mathematics," Addison-Wesley, 1989.
[2] S. Lang, "Undergraduate Analysis," 2nd ed., Springer-Verlag, 1997.