6. Vector Spaces

- Motivation

1. So many mathematical objects equipped with addition and scalar multiplication.
(e.g. $\mathbf{R}^n$, $\mathbf{C}^n$, $\mathcal{M}_{mn}$, $\mathcal{C}^0[a,b]$, $\cdots$)

2. So many properties that all such objects have in common.

3. Collect only a few common properties as axioms, and prove all the other properties as consequences of the axioms once and for all.

4. To study a new object with addition and scalar multiplication, you have only to verify the axioms. All the other properties are automatically available!

- Definition

A vector space V is a set with two operations $+$ and $\cdot$ satisfying the following properties. For $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and $k, p \in \mathbf{F}$

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\exists$ $\mathbf{0}$ s.t. $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for each $\mathbf{u}$.
4. For each $\mathbf{u}$, $\exists$ $-\mathbf{u}$ s.t. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $k \cdot (\mathbf{u} + \mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$, $(k + p) \cdot \mathbf{u} = k \cdot \mathbf{u} + p \cdot \mathbf{u}$
6. $(kp) \cdot \mathbf{u} = k \cdot (p \cdot \mathbf{u})$
7. $1 \cdot \mathbf{u} = \mathbf{u}$

Eg.

1. $\mathbf{F}^n$ (*i.e.* $\mathbf{R}^1, \mathbf{R}^2, \cdots, \mathbf{C}^1, \mathbf{C}^2, \cdots$)

2. $\mathcal{M}_{mn}$: *the set of all* $m \times n$ *matrices*

3. $\mathcal{P}$: *the set of all polynomials*

4. $\mathcal{F}[a, b]$: *the set of all real-valued functions on* $[a, b] \in \mathbf{R}$

   If $f(x) = \sin x$ *and* $g(x) = \cos x$, *then*

$$(f + g)(x) = f(x) + g(x) = \sin x + \cos x,$$

$$(3f)(x) = 3f(x) = 3 \sin x.$$

- Subspaces

If V is a vector space, a subset U of V is called a subspace of V if U is itself a vector space w.r.t. the operations in V.

Thm. *Let U be a nonempty subset of V. Then U is a subspace of V if and only if U is closed under the addition and scalar multiplication, i.e. $\mathbf{u} + \mathbf{v} \in U$ and $k\mathbf{u} \in U$ for any $\mathbf{u}, \mathbf{v} \in U$ and $k \in \mathbf{F}$.*

Cor. *Every subspace U of V contains the zero vector $\mathbf{0}$ of V.*

Eg. *The set* U *of all polynomials in* $\mathcal{P}$ *that have* 3 *as a root is a subspace of* $\mathcal{P}$*. Indeed, let* $p, q \in U$*. Then* $(p+q)(3) = p(3) + q(3) = 0$ *and so* $p + q \in U$*. Similarly,* $(kp)(3) = kp(3) = 0$ *and so* $kp \in U$*.*

Eg. *Let* $\mathcal{P}_n$ *be the set of all polynomials of degree at most* n*. Then* $\mathcal{P}_n$ *is a subspace of* $\mathcal{P}$ *for each* $n \geq 0$*.*

Eg. *The subset* $\mathcal{D}(a, b)$ *of all differentiable functions on* (a, b) *is a subspace of* $\mathcal{F}(a, b)$*.*

Eg. *We define*

$$\mathcal{C}^0(a,b) = \{f \in \mathcal{F}(a,b) \mid f \text{ is continuous } \},$$

$$\mathcal{C}^1(a,b) = \{f \in \mathcal{F}(a,b) \mid f' \text{ exists and continuous } \},$$

$$\mathcal{C}^2(a,b) = \{f \in \mathcal{F}(a,b) \mid f'' \text{ exists and continuous } \},$$

$$\vdots$$

$$\mathcal{C}^n(a,b) = \{f \in \mathcal{F}(a,b) \mid f^{(n)} \text{ exists and continuous } \},$$

$$\vdots$$

$$\mathcal{C}^\infty(a,b) = \{f \in \mathcal{F}(a,b) \mid f^{(n)} \text{ exists for all } n \}.$$

The sets $\mathcal{C}^0(a,b)$, $\mathcal{C}^1(a,b), \cdots, \mathcal{C}^\infty(a,b)$ *are subspaces of* $\mathcal{F}(a,b)$.

- Bases and Dimension

1. An expression $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v_n}$ is called a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$. The span of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$, denoted by $\mathrm{span}\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$, is the set of all linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$.

2. A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ is called linearly independent if $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ implies $a_1 = a_2 = \cdots = a_n = 0$.

3. A set B of vectors in a vector space V is called a basis of V if B is linearly independent and B spans V.

4. The number of vectors in a basis is called the dimension of V.

Eg. *Let* $p_1 = 1 + x + 4x^2$ *and* $p_2 = 1 + 5x + x^2$. *Determine whether* p_1 *and* p_2 *lie in* $\mathrm{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution.

$$\begin{aligned} p_1 &= s(1 + 2x - x^2) + t(3 + 5x + 2x^2) \\ &= (s + 3t) + (2s + 5t)x + (-2 + 2t)x^2 \end{aligned}$$

$$1 = s + 3t,\ 1 = 2s + 5t,\ 4 = -s + 2t.$$
$$s = -2, t = 1.$$

$$\begin{aligned} p_2 &= s(1 + 2x - x^2) + t(3 + 5x + 2x^2) \\ &= (s + 3t) + (2s + 5t)x + (-2 + 2t)x^2 \end{aligned}$$

$$1 = s + 3t,\ 5 = 2s + 5t,\ 1 = -s + 2t.$$

No solution!

Eg. *A set of polynomials of distinct degrees is linearly independent.*

Eg. *The set* $\{1, \sin x, \cos x\} \subseteq \mathcal{C}^0[-\pi, \pi]$ *is linearly independent. Actually,*

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots\} \subseteq \mathcal{C}^0[-\pi, \pi]$$

is linearly independent.

Eg. *The set* $\{1, x, x^2, \cdots, x^n\}$ *is a basis of* $\mathcal{P}_n$*. Thus* $\dim \mathcal{P}_n = n + 1$*.*

The set $\{1, x, x^2, \cdots\}$ *is a basis of* $\mathcal{P}$*. Thus* $\dim \mathcal{P}_n = \infty$*.*

Eg. *Show that* $\mathcal{P}_3 = \operatorname{span}\{x^2 + x^3, x, 1 + 2x^2, 3\}$.

Solution. *Since* $\{1, x, x^2, x^3\}$ *is a basis of* $\mathcal{P}_3$, *we have only to show*

$$1, x, x^2, x^3 \in \operatorname{span}\{x^2 + x^3, x, 1 + 2x^2, 3\}.$$

Thm. *Let* V *be a vector space and let* U *and* W *be subspaces of* V. *Then if* $U \subseteq W$ *and* $\dim U = \dim W$, *then* $U = W$.

Eg. *If a is a number, let W denote the subspace of all polynomials in $\mathcal{P}_n$ with a as a root, i.e.*

$$W = \{p \in \mathcal{P}_n | p(a) = 0\}.$$

Show that $\{(x-a), (x-a)^2, \cdots, (x-a)^n\}$ *is a basis of* W.

Solution. *Since the degrees are distinct,* $(x-a), (x-a)^2, \cdots, (x-a)^n$ *are linearly independent. Write*

$$U = \mathrm{span}\{(x-a), (x-a)^2, \cdots, (x-a)^n\}.$$

Then we have $U \subseteq W \subseteq \mathcal{P}_n$, $\dim U = n$, *and* $\dim \mathcal{P}_n = n+1$. *Hence* $n \leq \dim W \leq n+1$, *so* $\dim W = n$ *or* $\dim W = n+1$. *Then it follows from* **Thm** *that* $W = U$ *or* $W = \mathcal{P}_n$. *Since* $W \neq \mathcal{P}_n$, *we have* $W = U$.

Thm. *Every n-dimensional vector space is isomorphic to the space $\mathbf{F}^n$.*

Eg. $\mathcal{P}_n \simeq \mathbf{F}^{n+1}$

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \leftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$