3.3 Diagoanlization and Eigenvalues

- Eigenvalues and Eigenvectors

Def. *An $n \times n$ matrix D is called a diagonal matrix if D has the form*

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

Observations:

1. If $A = PBP^{-1}$ then $A^k = PB^kP^{-1}$ for $k = 1, 2, \cdots$.
2. If $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a diagonal matrix, then

$$D^k = \mathrm{diag}(\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k).$$

These observations suggest an efficient way to calculate A^k. More precisely, if we can find a diagonal matrix D s.t. $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$.

Def. A : $n \times n$ *matrix*
A number λ *is called an eigenvalue of* A *if* $\exists$ *a nonzero* $X \in \mathbf{F}^n$ *s.t.*

$$AX = \lambda X,$$

and such an X *is called an eigenvector of* A *corresponding to the eigenvalue* λ*, or* λ*-eigenvector for short.*

Eg. *If* $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ *and* $X = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$*, then* $AX = 4X$*. Thus* $\lambda = 4$ *is an eigenvalue of* A *and* $X = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ *is an eigenvector of* A *corresponding to* $\lambda = 4$*.*

Question : How can we get all the eigenvalues and eigenvectors of a matrix?

Def. A : $n \times n$ *matrix.*
The characteristic polynomial $c_A(x)$ *of* A *is defined by*

$$c_A(x) = \det(xI - A).$$

A characteristic polynomial is indeed a polynomial of degree n*.*

Thm. A : $n \times n$ *matrix.*

1. *The eigenvalues* λ *of* A *are the roots of* $c_A(x)$*.*
2. *The* λ*-eigenvectors* X *are the nonzero solutions of*

$$(\lambda I - A)X = O.$$

Proof. Assume that λ is an eigenvalue of A with an eigenvector X. Then $AX = \lambda X = \lambda IX$ and $(\lambda I - A)X = O$. Hence, X is a nontrivial solution of $(\lambda I - A)X = O$. Since the equation has a nontrivial solution, we have

$$\det(\lambda I - A) = c_A(\lambda) = 0$$

and so λ is a root of the characteristic polynomial.

Conversely, a similar argument shows that a root λ of $c_A(x)$ is indeed an eigenvalue and that a solution of $(\lambda I - A)X = O$ is an eigenvector. □

Eg. *Consider* $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$. *Then*

$$xI - A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix},$$

$$c_A(x) = \det(xI - A) = x^2 - 2x - 8 = (x-4)(x+2),$$

and so $\lambda_1 = 4$ *and* $\lambda_2 = -2$ *are eigenvalues.*

i) $\lambda_1 = 4$

$$(\lambda_1 I - A)X = \begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix} \, (t \neq 0).$$

ii) $\lambda_2 = -2$

$$\lambda_2 I - A = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} (t \neq 0).$$

Eg.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}, \quad xI - A = \begin{bmatrix} x-1 & -2 & 1 \\ -1 & x & -1 \\ -4 & 4 & x-5 \end{bmatrix},$$

$c_A(x) = \det(xI - A) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$,
and so $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ *are eigenvalues.*

i) $\lambda_1 = 1$

$$\lambda_1 I - A = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

ii) $\lambda_2 = 2$

$$\lambda_2 I - A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}.$$

iii) $\lambda_3 = 3$

$$\lambda_3 I - A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ -4 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}.$$

Eg.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad c_A(x) = \det(xI - A) = \begin{vmatrix} x & -1 \\ 1 & x \end{vmatrix} = x^2 + 1.$$

If $\mathbf{F} = \mathbf{R}$*,* A *has no eigenvalue.*
If $\mathbf{F} = \mathbf{C}$*,* A *has eigenvalues* $\pm i$*.*

Eg. *Consider an upper triangular matrix*

$$A = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

What are the eigenvalues of A?

Eg. A *and* A^T *have the same eigenvalues. Indeed,*

$$\begin{aligned} c_{A^T}(x) &= \det(xI - A^T) = \det(xI^T - A^T) \\ &= \det[(xI - A)^T] = \det(xI - A) = c_A(x). \end{aligned}$$

- Diagonalization

Def. *An $n \times n$ matrix A is called diagonalizable if $\exists$ an invertible P s.t. $P^{-1}AP = D$ is diagonal.*

Assume that $P^{-1}AP = D$. Write $P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ and $AP = PD$.

$$AP = A \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} = \begin{bmatrix} AX_1 & AX_2 & \cdots & AX_n \end{bmatrix}.$$

$$\begin{aligned} PD &= \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \end{bmatrix}. \end{aligned}$$

Comparing columns, we have

$$AX_i = \lambda_i X_i \text{ for each } i.$$

Thus the diagonal entries of D are eigenvalues of A, and the columns of P are corresponding eigenvectors.

Conversely, assume that we have $AX_i = \lambda_i X_i$ for each i with $P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ invertible. Write $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. It's easy to see that $AP = PD$ and $P^{-1}AP = D$.

Thm. A : $n \times n$ *matrix.*
A is diagonalizable if and only if it has eigenvectors $X_1, X_2, \cdots, X_n$ s.t. $P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ is invertible. In this case, $P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where λ_i is the eigenvalue of A corresponding to X_i.

Eg.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}, \quad \lambda_1 = 1,\ X_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix};$$

$$\lambda_2 = 2,\ X_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}; \quad \lambda_3 = 3,\ X_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

$$P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Eg.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \ c_A(x) = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix} = (x-2)(x+1)^2.$$

The eigenvalues are $\lambda_1 = 2$ *and* $\lambda_2 = -1$.

i) $\lambda_1 = 2$

$$\lambda_1 I - A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

ii) $\lambda_2 = -1$

$$\lambda_2 I - A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 = s \\ x_3 = t \end{cases}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Eg. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $c_A(x) = (x-1)^2$, *and so we have one eigenvalue* $\lambda = 1$.

$$\lambda I - A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A *is not diagonalizable.*

Eg. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. *Compute* A^{100}.

$$c_A(x) = \begin{bmatrix} x-1 & -2 \\ -3 & x-2 \end{bmatrix} = (x-4)(x+1).$$

The eigenvalues are $\lambda_1 = 4$ *and* $\lambda_2 = -1$.

i) $\lambda_1 = 4$

$$\lambda_1 I - A = \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

ii) $\lambda_2 = -1$

$$\lambda_2 I - A = \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}, \quad P^{-1}AP = D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4^{100} & 0 \\ 0 & 1^{100} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 \cdot 4^{100} & -1 \\ 3 \cdot 4^{100} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 \cdot 4^{100} + 3 & 2 \cdot 4^{100} - 2 \\ 3 \cdot 4^{100} - 3 & 3 \cdot 4^{100} + 2 \end{bmatrix}. \end{aligned}$$