3. Determinants and Diagonalization

3.1 The Laplace Expansion

- Determinants

Def. *Let* $X = \{1, 2, \cdots, n\}$. *A rearrangement of the elements of* X *is called a permutation of* X. *We denote the set of all permutations of* X *by* S_n. *Note that* $|S_n| = n!$.

Eg. *If* $X = \{1, 2, 3\}$, *then*

$$S_3 = \{123, 132, 213, 231, 312, 321\}.$$

A permutation $j_1 j_2 \cdots j_n$ is said to have an inversion if a larger j_r precedes a smaller j_s. A permutation is called even if the total number of inversions in it is even, or odd otherwise.

Eg. *The permutation $4312 \in S_4$ is odd.*

Def. *Let $A = [a_{ij}]$. We define*

$$\det(A) = |A| = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ of S_n. The sign is taken as $+$ or $-$ according to whether it is even or odd.

Eg.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

$$A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}, \quad \det(A) = 5 \cdot 3 - 4 \cdot 2 = 7.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \cdot 1 \cdot 2 + 2 \cdot 3 \cdot 3 + 3 \cdot 2 \cdot 1 \\ &\quad - 3 \cdot 1 \cdot 3 - 1 \cdot 3 \cdot 1 - 2 \cdot 2 \cdot 2 = 6. \end{aligned}$$

If A : 4×4, then $\det(A)$ has 24 terms.

If A : 5×5, then $\det(A)$ has 120 terms.

If A : 10×10, then $\det(A)$ has $3,628,800$ terms.

If A : 20×20, then $\det(A)$ has

$$2,432,902,008,176,640,000 \text{ terms.}$$

We need a practical definition.

- Cofactor Expansion

Def. $A : n \times n$ *matrix. Let* A_{ij} *be the* $(n-1) \times (n-1)$ *matrix obtained from* A *by deleting the row* i *and column* j*. The* (i,j)*-cofactor of* A *is defined to be*

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

Note that the sign is given by the pattern

$$\begin{bmatrix} + & - & + & - & + & - & + \cdots \\ - & + & - & + & - & + & - \cdots \\ + & - & + & - & + & - & + \cdots \\ - & + & - & + & - & + & - \cdots \\ + & - & + & - & + & - & + \cdots \\ & & & \cdots & & & \end{bmatrix}.$$

Eg.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} = 4, \quad C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} = 19.$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 34, \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 10.$$

Thm. [Laplace (Cofactor) Expansion - Stokes' Thm]
Let $A = [a_{ij}]$. For each i and j,

$$\begin{aligned} \det(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \end{aligned}$$

1. Choose a row (resp. a column).
2. Multiply each entry a_{ij} in the row (resp. the column) by the corresponding cofactor C_{ij}.
3. Add all the results.

Eg.

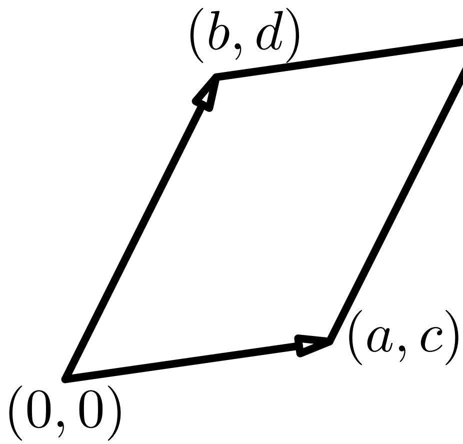
$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 1 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 3\left\{2\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ -2 & 1 \end{vmatrix}\right\}$$

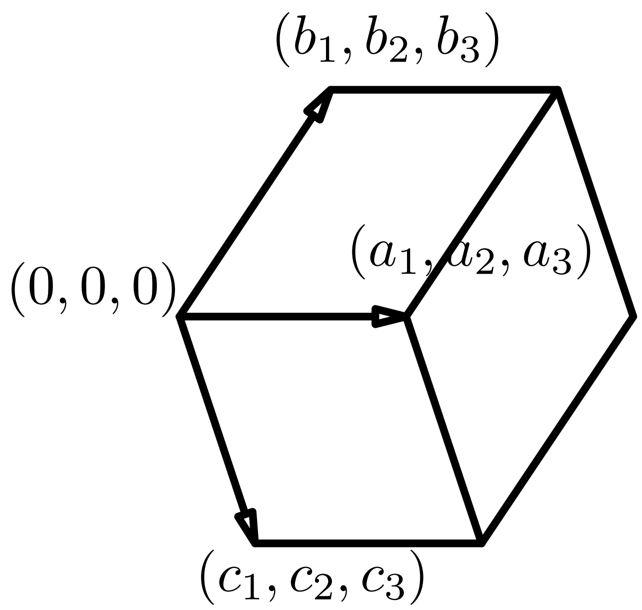
$$= 3(2\cdot 3 - 2\cdot 5) = -12.$$

- Geometric meaning of Determinant

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A)$ is the (signed) area of the parallelogram determined by (a, c) and (b, d) in $\mathbf{R}^2$.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (\pm) \text{ the area of the parallelogram}$$



$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (\pm) \text{ the volume of the parallelepiped in } \mathbf{R}^3$$

For A : $n \times n$, $\det(A)$ is the (signed) n-volume of the n-parallelepiped determined by the n vectors in $\mathbf{R}^n$.

- Properties of Determinant

Thm. *Assume that* $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$ *is an* $n \times n$ *matrix.*

1. *If* $R_i = 0$ *for some* i, *then* $\det(A) = 0$.
2. *If* $R_i = R_j$ *for some* $i \neq j$, *then* $\det(A) = 0$.
3. $\det \begin{bmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \end{bmatrix}$

4. $$\det \begin{bmatrix} \vdots \\ kR_i \\ \vdots \end{bmatrix} = k \det \begin{bmatrix} \vdots \\ R_i \\ \vdots \end{bmatrix},$$ *and so* $\det(kA) = k^n A$.

5. $$\det \begin{bmatrix} \vdots \\ R_i' + R_i'' \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R_i' \\ \vdots \end{bmatrix} + \det \begin{bmatrix} \vdots \\ R_i'' \\ \vdots \end{bmatrix}$$

6. $$\det \begin{bmatrix} \vdots \\ R_i + kR_j \\ \vdots \\ R_j \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{bmatrix}$$

Thm. *Assume that* $A = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$.

1. *If* $C_i = 0$ *for some* i, *then* $\det(A) = 0$.

2. *If* $C_i = C_j$ *for some* $i \neq j$, *then* $\det(A) = 0$.

3. $\det \begin{bmatrix} \cdots & C_i & \cdots & C_j & \cdots \end{bmatrix} = -\det \begin{bmatrix} \cdots & C_j & \cdots & C_i & \cdots \end{bmatrix}$

4. $\det \begin{bmatrix} \cdots & kC_i & \cdots \end{bmatrix} = k \det \begin{bmatrix} \cdots & C_i & \cdots \end{bmatrix}$,

   and so $\det(kA) = k^n A$.

5. $\det \begin{bmatrix} \cdots & C_i' + C_i'' & \cdots \end{bmatrix} = \det \begin{bmatrix} \cdots & C_i' & \cdots \end{bmatrix} + \det \begin{bmatrix} \cdots & C_i'' & \cdots \end{bmatrix}$

6. $\det \begin{bmatrix} \cdots & C_i + kC_j & \cdots & C_j & \cdots \end{bmatrix} = \det \begin{bmatrix} \cdots & C_i & \cdots & C_j & \cdots \end{bmatrix}$

Eg.

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 3 & -1 & 5 \\ 2 & 8 & 7 \\ 1 & 2 & -1 \end{vmatrix} = - \begin{vmatrix} 5 & -1 & 3 \\ 7 & 8 & 2 \\ -1 & 2 & 1 \end{vmatrix},$$

$$\begin{vmatrix} 8 & 1 & 2 \\ 3 & 0 & 9 \\ 1 & 2 & -1 \end{vmatrix} = 3 \begin{vmatrix} 8 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix}, \qquad \begin{vmatrix} 2 & 1 & 2 \\ 4 & 0 & 4 \\ 1 & 3 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 2 & 5 & 2 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 9 & 20 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix}.$$

Eg. *The Vandermonde determinant*

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i<j} (x_j - x_i).$$

Thm. *If A is a triangular matrix, then $\det(A)$ is the product of entries of the main diagonal.*

Proof. Use the cofactor expansion. □

Thm. *Consider $\begin{bmatrix} A & X \\ O & B \end{bmatrix}$ and $\begin{bmatrix} A & O \\ Y & B \end{bmatrix}$ in block form, where A and B are square matrices. Then*

$$\det \begin{bmatrix} A & X \\ O & B \end{bmatrix} = \det A \det B$$

and

$$\det \begin{bmatrix} A & O \\ Y & B \end{bmatrix} = \det A \det B.$$

Eg.

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 3 & 0 & 8 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 8 \\ 1 & -1 \end{vmatrix} = -11.$$

$$\begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{vmatrix} = -1.$$

$$\begin{vmatrix} 1 & 2 & -1 & 10 \\ 3 & 4 & 3 & -7 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = (-2)(-2) = 4.$$

3.2 Determinants and Matrix Inverses

Thm.

$$\det(AB) = \det A \det B$$

Proof. It is easy to see that $\det(EA) = \det E \det A$ for any elementary matrix E. Thus

$$\det(E_1 E_2 \cdots E_k A) = \det E_1 \det E_2 \cdots \det E_k \det A.$$

If $A \rightsquigarrow R$ in r.r.e.f., then $R = E_k \cdots E_1 A$ and $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} R$. Note that E_i^{-1} is an elementary matrix for each i.

$$\begin{aligned}\det(AB) &= \det(E_1^{-1}E_2^{-1}\cdots E_k^{-1}RB)\\ &= \det(E_1^{-1})\cdots\det(E_k^{-1})\det(RB).\end{aligned}$$

$$\begin{aligned}(\det A)(\det B) &= \det(E_1^{-1}E_2^{-1}\cdots E_k^{-1}R)\det B\\ &= \det(E_1^{-1})\cdots\det(E_k^{-1})\det R\det B.\end{aligned}$$

We have only to show that $\det(RB) = \det R \det B$. Note that either $R = I$ or R has a row of zeros. If $R = I$, then $\det(RB) = \det B = \det R \det B$. If R has a row of zeros, then RB also has a row of zeros. Thus $\det(RB) = 0 = \det R \det B$. □

Cor.

$$\det(A_1 A_2 \cdots A_{k-1} A_k) = \det(A_1)\det(A_2)\cdots\det(A_{k-1})\det(A_k),$$

$$\det(A^k) = (\det A)^k.$$

Thm. *A is invertible $\Leftrightarrow \det A \neq 0$. In this case,*

$$\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det A}.$$

Proof. If A is invertible, then $AA^{-1} = I$ and

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}.$$

Hence, $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$.

Conversely, assume that $\det A \neq 0$. If $A \rightsquigarrow R$ in r.r.e.f. and $R = E_k \cdots E_1 A$, then either $R = I$ or R has a row of zeros. Note that $\det E \neq 0$ for every elementary matrix E. Thus

$$\begin{aligned} 0 &\neq \det E_k \cdots \det E_1 \det A \\ &= \det(E_k \cdots E_1 A) = \det R. \end{aligned}$$

Therefore, R cannot have a row of zeros, so $R = I$. □

Eg.

$$A = \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & -3 & 8 & 3 \\ -2 & 7 & -1 & 0 \\ 4 & 0 & 6 & 2 \end{bmatrix}, \quad \det A = 36 \neq 0.$$

Thus A is invertible.

Eg. *AB is invertible $\Leftrightarrow$ A and B are invertible.*

Thm.

$$\det(A^T) = \det A$$

Proof. It is easy to see that $\det E^T = \det E$ for every elementary matrix E. If A is not invertible, then neither is A^T; so $\det A = 0 = \det A^T$. If A is invertible, $A = E_k \cdots E_1$ and $A^T = E_1^T \cdots E_k^T$.

$$\begin{aligned} \det A^T &= \det E_1^T \cdots \det E_k^T \\ &= \det E_1 \cdots \det E_k \\ &= \det E_k \cdots \det E_1 = \det A. \end{aligned}$$

□

Eg. *A matrix is called orthogonal if* $A^{-1} = A^T$. *If* A *is orthogonal, then* $AA^{-1} = AA^T = I$ *and* $1 = \det(AA^T) = \det A \det A^T = (\det A)^2$. *Hence,* $\det A = \pm 1$.

- Classical Adjoint

Def. *The classical adjoint of* A*, denoted by* $\mathrm{adj}(A)$*, is the transpose of the cofactor matrix, i.e.*

$$\mathrm{adj}(A) = [C_{ij}]^T.$$

Eg.

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix}, \qquad \mathrm{adj}(A) = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}.$$

Lem. *Let* $A = [a_{ij}]$*.*

$$\begin{aligned} a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} &= 0 \text{ for } i \neq k \\ a_{1j}C_{1k} + a_{2j}C_{2k} + \cdots + a_{nj}C_{nk} &= 0 \text{ for } j \neq k. \end{aligned}$$

Proof. Let

$$A = \begin{bmatrix} C_1 & \cdots & C_i & \cdots & C_k & \cdots & C_n \end{bmatrix}$$

and

$$B = \begin{bmatrix} C_1 & \cdots & C_i & \cdots & C_i & \cdots & C_n \end{bmatrix}.$$

Then

$$0 = \det B = a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn}.$$

□

Thm.

$$A(\mathrm{adj}(A)) = (\mathrm{adj}(A))A = (\det A)I.$$

If $\det A \neq 0$, *then* $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$.

Proof. Write $\mathrm{adj}(A) = [d_{ij}]$, and so $d_{ij} = C_{ji}$. Let $X = A(\mathrm{adj}(A)) = [x_{ij}]$.

$$x_{ij} = \sum_{k=1}^{n} a_{ik}d_{kj} = \sum_{k=1}^{n} a_{ik}C_{jk} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

□

Rmk. *The formula is not practical.*

Eg.

$$\det(\operatorname{adj}(A)) = (\det A)^{n-1}$$

Indeed, it follows from $A(\operatorname{adj}(A)) = (\det A)I$ *that*

$$\det A \det(\operatorname{adj}(A)) = (\det A)^n.$$

If $\det A \neq 0$ *then divide it by* $\det A$.

If $\det A = 0$, *then* $A(\operatorname{adj}(A)) = O$. *If* $\operatorname{adj}(A)$ *is invertible, then* $A = O$, *so* $\operatorname{adj}(A) = O$, *a contradiction. Thus* $\operatorname{adj}(A)$ *is not invertible and* $\det(\operatorname{adj}(A)) = 0$.

- Cramer's Rule

Consider $AX = B$. If A is invertible, $X = A^{-1}B$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

$$x_i = \frac{1}{\det A}(b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni})$$

Write

$$A = \begin{bmatrix} C_1 & \cdots & C_i & \cdots & C_n \end{bmatrix}$$

and let

$$A_i = \begin{bmatrix} C_1 & \cdots & C_{i-1} & B & C_{i+1} & \cdots & C_n \end{bmatrix}.$$

From the cofactor expansion, we have

$$x_i = \frac{1}{\det A}(b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni}) = \frac{\det A_i}{\det A}.$$

Thm. [Cramer's Rule] *If A is invertible, then the solution of the system $AX = B$ is given by*

$$X = \frac{1}{\det A}\begin{bmatrix} \det A_1 \\ \det A_2 \\ \vdots \\ \det A_n \end{bmatrix},$$

where A_i is the matrix obtained from A by replacing ith column by B.

Eg.

$$\begin{cases} -2x_1 + 3x_2 - x_3 = 1 \\ x_1 + 2x_2 - x_3 = 4 \\ -2x_1 - x_2 + x_3 = -3 \end{cases}$$

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2, \quad |A_1| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4,$$

$$|A_2| = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -6, \quad |A_3| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = -8.$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Rmk. *Do you think it is practical?*