5.2 Rank of Matrix

- Row Space and Column Space

  Let A be an $m \times n$ matrix.

  - the row space of A = the span of rows of $A \subset \mathbf{F}^n$
    $=$ row$A$
  - the column space of A = the span of columns of $A \subset \mathbf{F}^m$
    $=$ col$A$

Thm. $A : m \times n,\ U : p \times m,\ V : n \times q$

1. *col$(AV) \subset$ col$A$. If V is invertible, col$(AV) =$ col$A$.*
2. *row$(UA) \subset$ row$A$. If U is invertible, row$(UA) =$ row$A$.*

Proof.

$$V = \begin{bmatrix} V_1 & \cdots & V_q \end{bmatrix}, \quad AV = \begin{bmatrix} AV_1 \cdots AV_q \end{bmatrix}$$

$$A = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}, \; V_j = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{aligned} AV_j &= \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= v_1 C_1 + v_2 C_2 + \cdots v_n C_n \in \mathsf{col} A \end{aligned}$$

$$\mathsf{col}(AV) = \mathsf{span}\{AV_1, \cdots, AV_q\} \subset \mathsf{col} A$$

If V is invertible,

$$\text{col}A = \text{col}(AVV^{-1}) \subset \text{col}(AV).$$

Hence, $\text{col}A = \text{col}(AV)$.

$$\text{col}[(UA)^T] = \text{col}[A^TU^T] \subset \text{col}(A^T) \Leftrightarrow \text{row}(UA) \subset \text{row}A.$$

If U is invertible, U^T is invertible.

$$\text{col}[(UA)^T] = \text{col}[A^TU^T] = \text{col}(A^T) \Leftrightarrow \text{row}(UA) = \text{row}A.$$

$\square$

Fact. *Assume that $A \rightsquigarrow R$ in r.r.e.f.*

1. *The nonzero rows of R form a basis of* row$R$.
2. *The columns of R containing leading 1's form a basis of* col$R$.

Thm. [Rank Thm]

$$\dim(\textit{row}A) = \dim(\textit{col}A) = \textit{rank}A.$$

Moreover, suppose A is carried to R in r.r.e.f.

1. *The nonzero rows of R form a basis of* row$A$.
2. *The columns of A corresponding leading 1's of R form a basis of* col$A$.

Proof. As usual, $E_k \cdots E_1 A = R$, $U = E_k \cdots E_1$ is invertible, and $UA = R$.

$$\text{row}R = \text{row}(UA) = \text{row}A.$$

By Fact 1, we obtain the first part of the theorem.

Write $A = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}$.

$$R = UA = U \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{bmatrix} UC_1 & \cdots & UC_n \end{bmatrix}$$

By Fact 2, the columns $UC_{j_1}, \cdots, UC_{j_r}$ of R containing leading 1's form a basis of col$R$. It is easy to see that $\{C_{j_1}, \cdots, C_{j_r}\}$ is linearly independent.

Similarly,

$$\begin{aligned} UC_j &= a_1UC_{j_1} + a_2UC_{j_2} + \cdots + a_rUC_{j_r} \\ &= U(a_1C_{j_1} + a_2C_{j_2} + \cdots + a_rC_{j_r}). \end{aligned}$$

Thus, $C_j = a_1C_{j_1} + a_2C_{j_2} + \cdots + a_rC_{j_r}$, and

$$\{C_1, C_2, \cdots, C_n\} \subset \text{span}\{C_{j_1}, \cdots, C_{j_r}\}.$$

Therefore, we have

$$\dim \text{row}A = \dim \text{col}A = r.$$

□

Eg.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1. *rank*$A = 2$,

2. *a basis of row*$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$,

3. *a basis of col*$A = \left\{ \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Cor.

1. rank$A$ = rankA^T.
2. *If* A : $m \times n$, *then* rank$A \leq \min(m, n)$.
3. rank$A$ = rank(UA) = rank(AV) *where* U *and* V *are invertible.*
4. A *is invertible if and only if* rank$A = n$.

- A Basis from a Spanning Set

Eg. *Find a basis for* $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 4 \\ -3 \end{bmatrix} \right\}$.

1.

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $W = \text{row}A$ *and a basis of* W *is given by*

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

2.

$$B = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & -\frac{49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $W = \text{col}A$ *and another basis of* W *is given by*

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

Eg.

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1, x_2, x_4 \ : \ \textit{leading variables} \\ x_3 = r, \ x_5 = s \ : \ \textit{parameters} \end{cases}$$

$$X = \begin{bmatrix} -2r - s \\ -2r + s \\ r \\ -2s \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$: *basic solutions*

Thm. *Let* A *:* $m \times n$ *with rank* r*.*

1. *If* $X_1, X_2, \cdots, X_{n-r}$ *are basic solutions of* $AX = O$*, then* $\{X_1, \cdots, X_{n-r}\}$ *is a basis of* null$A$ $(= \ker A)$*, and we have*

$$\dim(\text{null}A) = n - r.$$

2. im$A$ = col$A$*, and*

$$\dim(\text{im}A) = r = \text{rank}A.$$

Hence,

$$n = \dim(\text{null}A) + \dim(\text{im}A).$$

Thm. *Assume that* $A : m \times n$. *Then*
A *has rank* r *if and only if* $\exists$ *invertible* U *and* V *s.t.*

$$UAV = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix},$$

where I_r : $r \times r$ *identity matrix.*

Proof. Assume A has rank r. Transform A into r.r.e.f. $UA = R$, and then transform R^T into r.r.e.f.

$$V'R^T = V'A^TU^T = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

Note that R and R^T have rank r. Letting $V' = V^T$, we obtain

$$UAV = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

□

– Algorithm for obtaining U and V

$$\begin{bmatrix} A & I \end{bmatrix} \Rightarrow U \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} R & U \end{bmatrix}$$

$$\begin{bmatrix} R^T & I \end{bmatrix} \Rightarrow V^T \begin{bmatrix} R^T & I \end{bmatrix} = \begin{bmatrix} \begin{matrix} I_r & O \\ O & O \end{matrix} & V^T \end{bmatrix}$$

Eg.

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$$

Find U *and* V *s.t.* $UAV = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ *with* $r =$ *rank*A.

$$\begin{bmatrix} A & I \end{bmatrix} \Rightarrow \begin{bmatrix} R & U \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R^T & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3 & 5 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -5 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & -5 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Def. *Assume that A is given. We define*

$$\overline{A} = [\,\overline{a_{ij}}\,], \text{ the conjugate of } A,$$

and

$$A^* = \overline{A^T}, \text{ the adjoint of } A.$$

Eg.

$$A = \begin{bmatrix} 4+i & -2+3i \\ 6+4i & 3 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} 4-i & -2-3i \\ 6-4i & 3 \end{bmatrix}, \; A^* = \begin{bmatrix} 4-i & 6-4i \\ -2-3i & 3 \end{bmatrix}.$$

Prop.

1. $A^* = \overline{A^T} = (\overline{A})^T$

2. $(AB)^* = B^* A^*$

Def. *We define*

A : *symmetric* $\Leftrightarrow$ $A^T = A$

A : *Hermitian* $\Leftrightarrow$ $A^* = A$

Thm. *Let A be an $m \times n$ matrix. T.F.A.E.*

1. $AX = O$ *has only the trivial solution.*

2. *The columns of A are linearly independent.*

3. *rank*$A = n$

4. A^*A *is invertible.*

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3$: easy.

$4 \Rightarrow 1$: If $AX = O$, then $A^*AX = O$ so $X = O$.

$1 \Rightarrow 4$: It suffices to show $(A^*A)X = O$ has only the trivial solution. Assume that $(A^*A)X = O$. Write $AX = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $(AX)^* = \begin{bmatrix} \overline{y_1} & \cdots & \overline{y_n} \end{bmatrix}$.

$$\begin{aligned} 0 &= X^*A^*AX = (AX)^*AX \\ &= \begin{bmatrix} \overline{y_1} & \cdots & \overline{y_n} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = |y_1|^2 + |y_2|^2 + \cdots + |y_n|^2 \end{aligned}$$

Thus $y_1 = y_2 = \cdots = y_n = 0$, and $AX = O$. By assumption we have $X = O$. $\square$

Thm. *Let A be an $m \times n$ matrix. T.F.A.E.*

1. $AX = B$ *has a solution for every* B.

2. *The columns of* A *span* $\mathbf{F}^n$.

3. *rank*$A = m$

4. AA^* *is invertible.*