4.1 Vectors and Lines

- Definition

  – scalar : magnitude
  vector : magnitude and direction

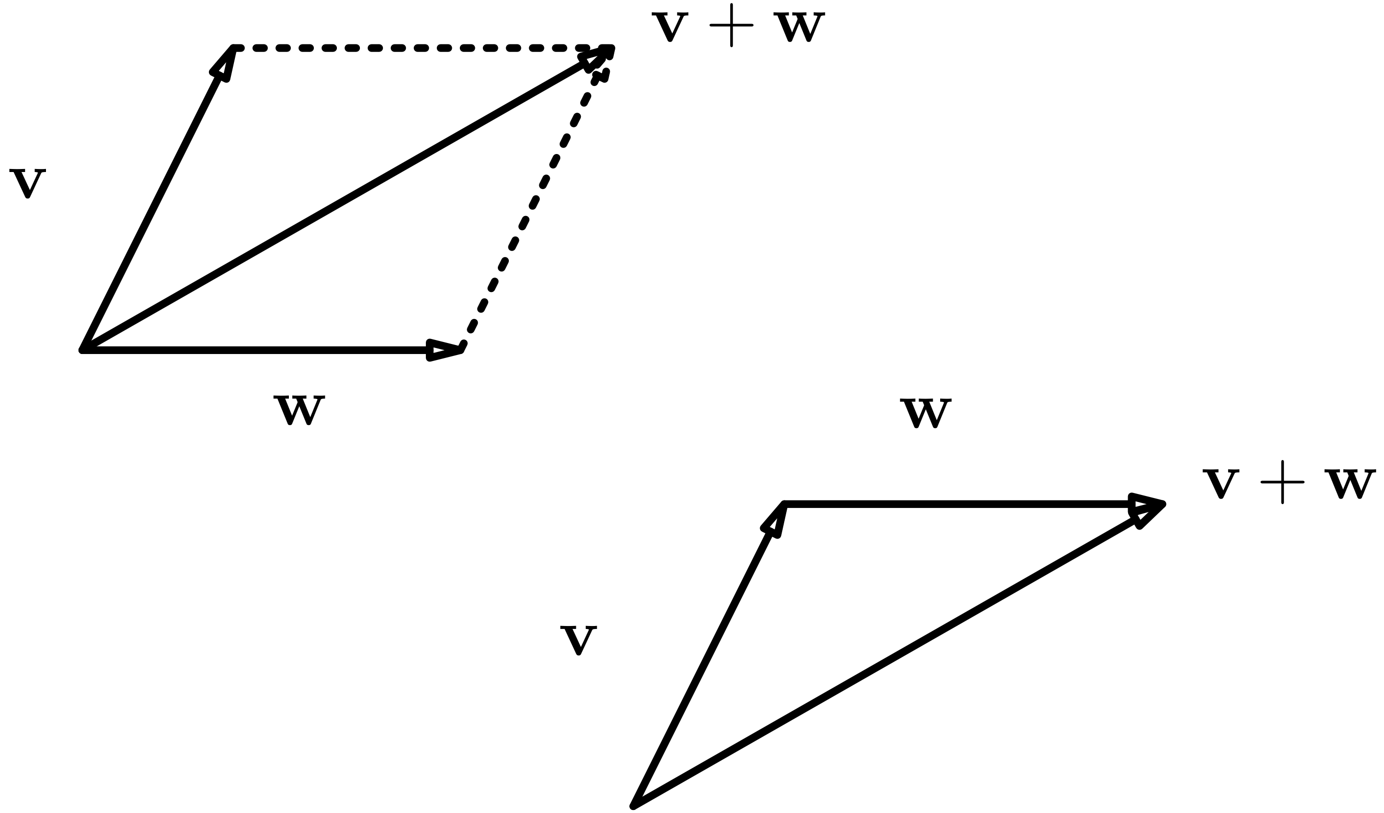
  Geometrically, a vector $\mathbf{v}$ can be represented by an arrow.
  We denote the length of $\mathbf{v}$ by $\|\mathbf{v}\|$.

  – zero vector $\mathbf{0}$: $\|\mathbf{0}\| = \mathbf{0}$

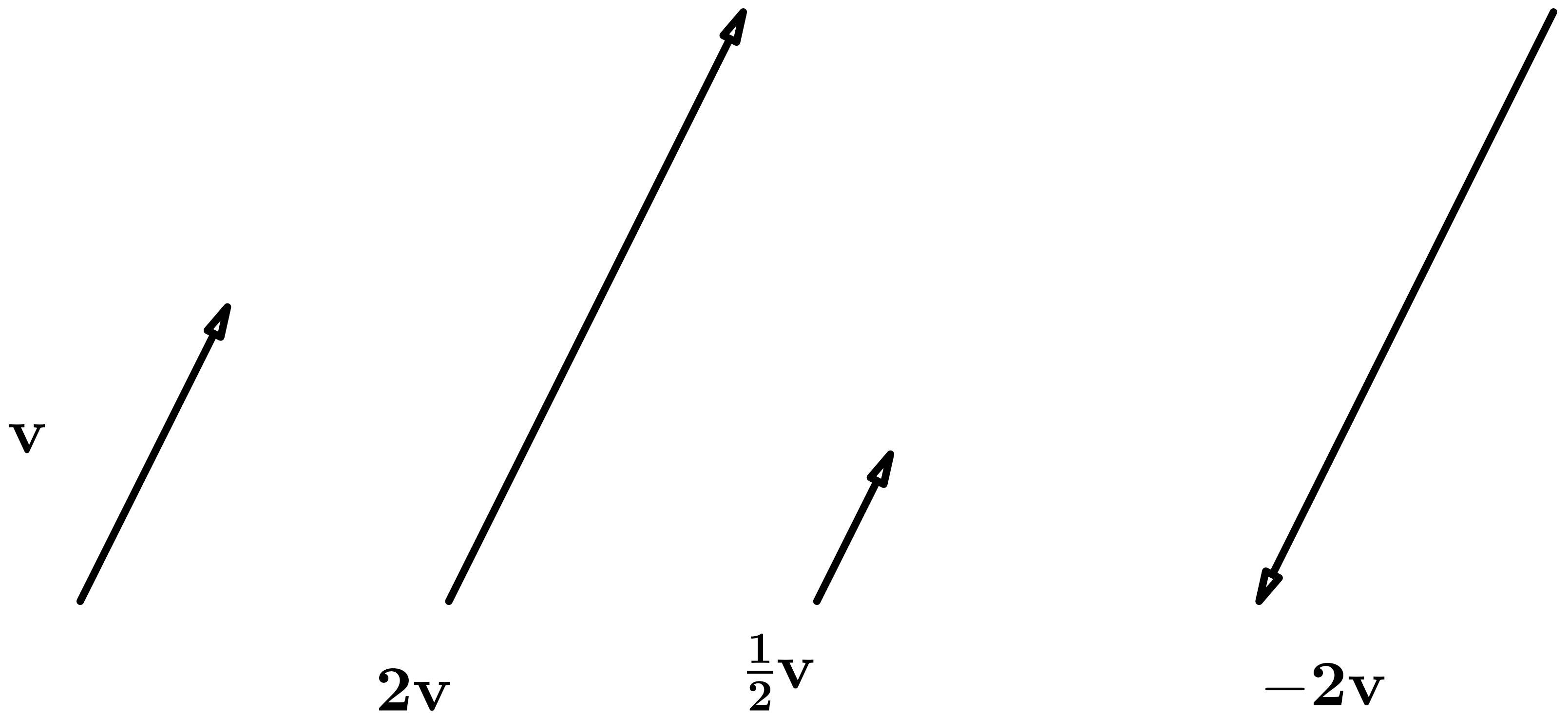
  – Given $\mathbf{v}$, we have the negative $-\mathbf{v}$.

  – $\mathbf{v} = \mathbf{w}$ if the same length and the same direction

– sum $\mathbf{v} + \mathbf{w}$



- scalar multiplication $a\mathbf{v}$ $(a \in \mathbf{R})$



- subtraction $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

Thm. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$: *vectors,* $k, p \in \mathbf{R}$

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\exists\ \mathbf{0}$ *s.t.* $\mathbf{0} + \mathbf{u} = \mathbf{u}$ *for each* $\mathbf{u}$.
4. *For each* $\mathbf{u}$, $\exists\ -\mathbf{u}$ *s.t.* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$, $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
6. $(kp)\mathbf{u} = k(p\mathbf{u})$,
7. $1 \cdot \mathbf{u} = \mathbf{u}$

Thm. A, B, C : *matrices of the same size,* $k, p \in \mathbf{F}$

1. $A + B = B + A$,
2. $A + (B + C) = (A + B) + C$
3. $\exists\ O$ *s.t.* $O + A = A$ *for each* A.
4. *For each* A, $\exists\ -A$ *s.t.* $A + (-A) = O$.
5. $k(A + B) = kA + kB$, $(k + p)A = kA + pA$
6. $(kp)A = k(pA)$,
7. $1 \cdot A = A$

Thm. f, g, h : *continuous functions on* D, $k, p \in \mathbf{F}$

1. $f + g = g + f$,
2. $f + (g + h) = (f + g) + h$
3. $\exists\ 0$ *s.t.* $0 + f = f$ *for each* f.
4. *For each* f, $\exists\ -f$ *s.t.* $f + (-f) = 0$.
5. $k(f + g) = kf + kg$, $(k + p)f = kf + pf$
6. $(kp)f = k(pf)$,
7. $1 \cdot f = f$

The notion of vector space!

1. The set of matrices of the same size
2. The set of vectors in $\mathbf{R}^3$
3. The set of continuous functions on D
4. ...
5. ...

and so on.

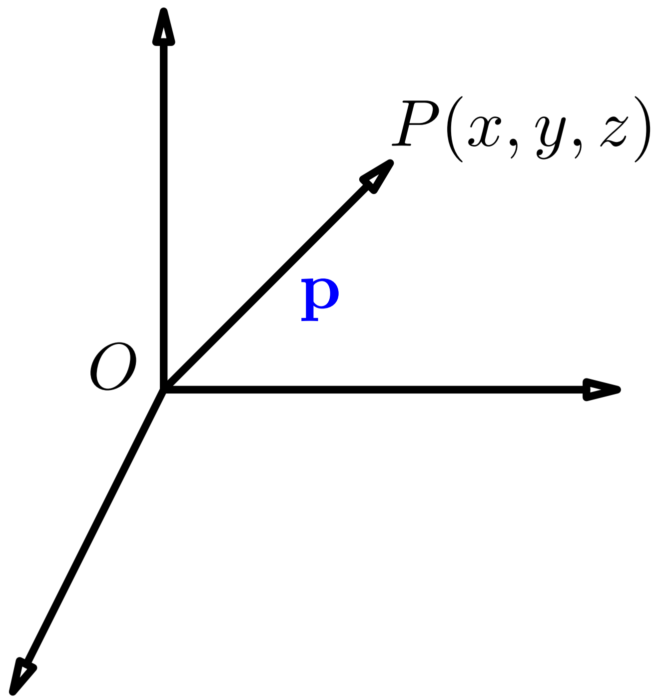
The theorem says that we can manipulate vectors as if they are variables w.r.t. addition and scalar multiplication.

Eg.

$$\begin{aligned} 5(\mathbf{u} - 2\mathbf{v}) + 6(5\mathbf{u} + 2\mathbf{v}) &= 5\mathbf{u} - 10\mathbf{v} + 30\mathbf{u} + 12\mathbf{v} \\ &= 35\mathbf{u} + 2\mathbf{v}. \end{aligned}$$

- Coordinates

Consider a point $P = (x, y, z)$. Then we obtain a vector $\mathbf{p} = \overrightarrow{OP}$: the position vector. Conversely, a vector $\mathbf{p}$ determines a unique point P. Thus we identify each point with the corresponding position vector.



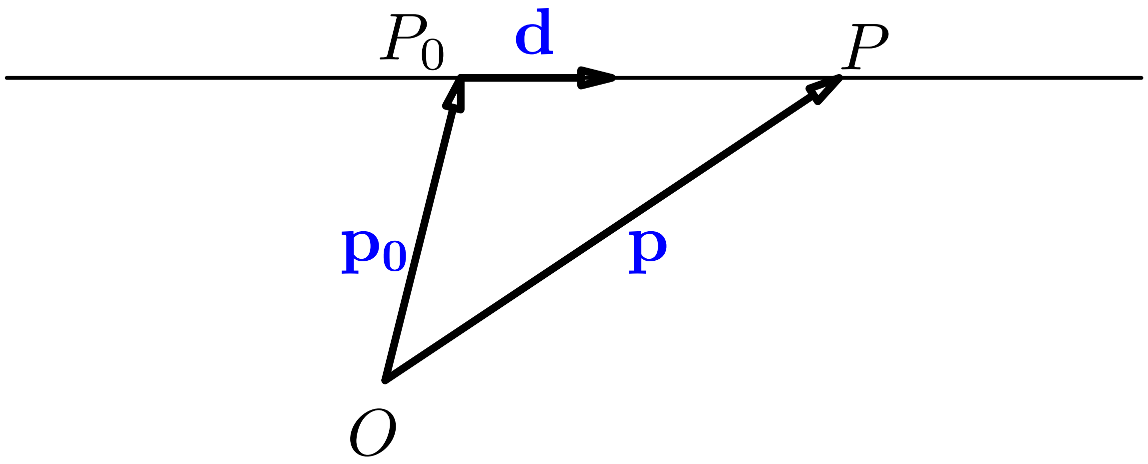
Given $\mathbf{u} = (x, y, z)$ and $\mathbf{u}_1 = (x_1, y_1, z_1)$, we have

$$\mathbf{u} + \mathbf{u}_1 = (x + x_1, y + y_1, z + z_1),$$

$$a\mathbf{u} = (ax, ay, az),$$

$$\mathbf{u} - \mathbf{u}_1 = (x - x_1, y - y_1, z - z_1).$$

- Lines



Assume that $\mathbf{p}_0$ and $\mathbf{d}$ are given. Then $\mathbf{p}$ is the position vector of a point P on the line if and only if

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \qquad (t \in \mathbf{R}).$$

If $\mathbf{p} = (x, y, z)$, $\mathbf{d} = (a, b, c)$, $\mathbf{p}_0 = (x_0, y_0, z_0)$, then we have

$$\begin{cases} x = x_0 + ta, \\ y = y_0 + tb, \\ z = z_0 + tc, \end{cases} \qquad (t \in \mathbf{R}).$$

This is the equation of the line through $\mathbf{p}_0$ parallel to $\mathbf{d}$.

- Planes

Later ... we need the notion of inner product and cross product of vectors.

5.1 Subspaces and Dimension

- Subspaces of $\mathbf{F}^n$

vector $=$ point in $\mathbf{R}^3 \leftrightarrow (x, y, z)$ coordinates

????? $\leftrightarrow (a_1, a_2, \cdots, a_n)$

$$\begin{aligned} \mathbf{R}^n &= \{(a_1, a_2, \cdots, a_n) | a_i \in \mathbf{R}\} \\ &\cong \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_i \in \mathbf{R} \right\} \end{aligned}$$

$$\mathbf{C}^n = \{(a_1, a_2, \cdots, a_n) | a_i \in \mathbf{C}\}$$
$$\cong \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_i \in \mathbf{C} \right\}$$

$\mathbf{F}^n = \mathbf{R}^n$ or $\mathbf{C}^n$

The n-tuples in $\mathbf{F}^n$ will be called vectors.

- Subspaces

A subset U of $\mathbf{F}^n$ is called a subspace if it satisfies the following conditions.

1. If $X, Y \in U$, then $X + Y \in U$.
2. If $X \in U$, then $rX \in U$ for $r \in \mathbf{F}$.

Eg.

1. $\mathbf{F}^n$

2. $\{0\}$ *: the zero subspace*

3. *a line through the origin in* $\mathbf{R}^n$: $\{t\mathbf{d}\}$

If $t_1\mathbf{d}$ *and* $t_2\mathbf{d}$ *on the line, then* $t_1\mathbf{d} + t_2\mathbf{d} = (t_1 + t_2)\mathbf{d}$ *and* $r(t_1\mathbf{d}) = (rt_1)\mathbf{d}$.

4. *Let* A *be an* $m \times n$ *matrix. We define*

$$nullA = kerA = \{X \in \mathbf{F}^n | AX = O\} \text{ and}$$

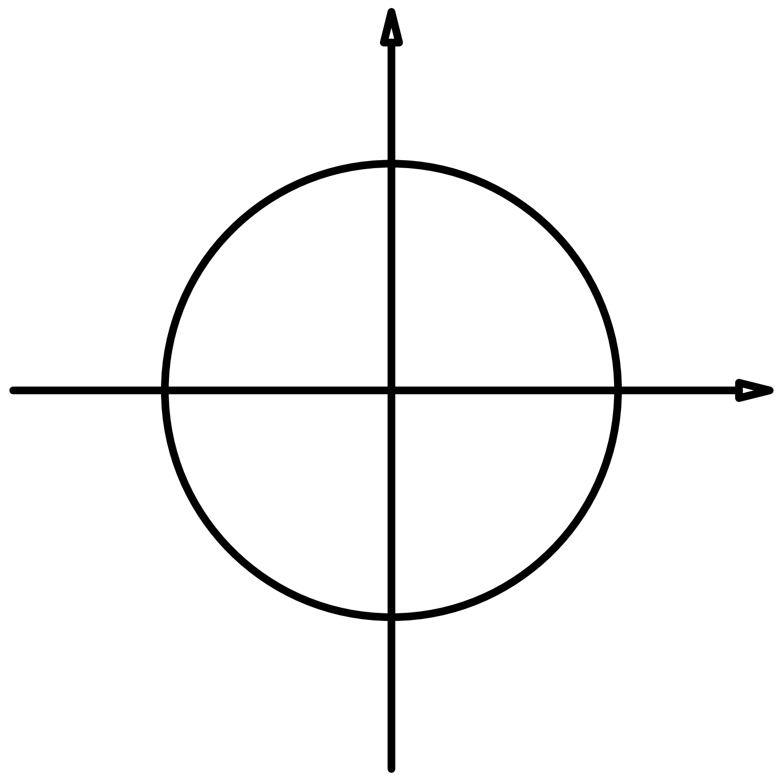
$$imA = \{Y \in \mathbf{F}^m | Y = AX \text{ for some } X \in \mathbf{F}^n\}.$$

If $X_1, X_2 \in kerA$, *then* $A(X_1 + X_2) = AX_1 + AX_2 = O$ *and* $A(rX_1) = r(AX_1) = O$. *If* $Y_1, Y_2 \in imA$, *then* $\exists\ X_1, X_2$ *s.t.* $AX_1 = Y_1$ *and* $AX_2 = Y_2$. *Now* $A(X_1 + X_2) = Y_1 + Y_2$ *and* $A(rX_1) = rY_1$.

5.

$$U = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}.$$

We have $(1, 0), (0, 1) \in U$, *but* $(1, 0) + (0, 1) = (1, 1) \notin U$.
Thus U *is not a subspace of* $\mathbf{R}^2$.



- Spanning sets

Def. *Assume that* $X_1, X_2, \cdots, X_k \in \mathbf{F}^n$. *An expression*

$$a_1X_1 + a_2X_2 + \cdots + a_kX_k$$

is called a linear combination *of* $X_1, X_2, \cdots, X_k$ *(*$a_i \in \mathbf{F}$*).*

The span of $X_1, X_2, \cdots, X_k$ *is the set of all linear combinations of* $X_1, X_2, \cdots, X_k$.

$$span\{X_1, X_2, \cdots, X_k\} = \{a_1X_1 + a_2X_2 + \cdots + a_kX_k | a_i \in \mathbf{F}\}$$

Thm. *Assume that* $X_1, X_2, \cdots, X_k \in \mathbf{F}^n$.

1. The $span\{X_1, X_2, \cdots, X_k\}$ *is a subspace of* $\mathbf{F}^n$.

2. If W *is a subspace containing* $X_1, X_2, \cdots, X_k$, *then*

$$span\{X_1, X_2, \cdots, X_k\} \subset W.$$

Proof. 1. Let $U = span\{X_1, X_2, \cdots, X_k\}$. If

$$Y = s_1X_1 + \cdots + s_kX_k,\ Z = t_1X_1 + \cdots + t_kX_k \in U,$$

then $Y + Z = (s_1 + t_1)X_1 + \cdots + (s_k + t_k)X_k \in U$ and $rY = rs_1X_1 + \cdots + rs_kX_k \in U$.

2. Clear! □

The $span\{X_1, X_2, \cdots, X_k\}$ is the smallest subspace containing $X_1, \cdots, X_k$.

If $U = span\{X_1, X_2, \cdots, X_k\}$, then $\{X_1, X_2, \cdots, X_k\}$ is a spanning set of U, and U is spanned by the X_i's.

Eg. *Recall*

Thm. *Given $AX = O$, every solution is a linear combination of the basic solutions.*

Equivalently, the $ker A$ is the span of the basic solutions.

Assume $A = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$: $m \times n$ matrix. Then

$$imA = span\{C_1, C_2, \cdots, C_n\}.$$

Proof. For $X \in \mathbf{F}^n$,

$$AX = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1C_1 + x_2C_2 + \cdots + x_nC_n$$

$$\begin{aligned} imA &= \{AX | X \in \mathbf{F}^n\} = \{x_1C_1 + x_2C_2 + \cdots + x_nC_n\} \\ &= span\{C_1, C_2, \cdots, C_n\} \end{aligned}$$

□

- Independence

Def. $\{X_1, X_2, \cdots, X_k\}$: *linearly independent* *if* $t_1X_1 + t_2X_2 + \cdots + t_kX_k = 0$ *implies* $t_1 = t_2 = \cdots = t_k = 0$.

Thm. *If* $\{X_1, X_2, \cdots, X_k\}$ *is linearly independent,* $X \in span\{X_1, X_2, \cdots, X_k\}$ *has a unique representation as a linear combination of the* X_i*'s.*

Proof.

$$r_1X_1 + \cdots + r_kX_k = s_1X_1 + \cdots + s_kX_k$$

$$(r_1 - s_1)X_1 + \cdots + (r_k - s_k)X_k = 0$$

Thus we have $r_i = s_i$ for all i. □

Eg. $X_1, X_2, X_1 + X_2$

$$2X_1 + 2X_2 = 2(X_1 + X_2)$$

Eg.

$$\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\3\end{bmatrix} \qquad r\begin{bmatrix}1\\0\\1\\2\end{bmatrix} + s\begin{bmatrix}0\\1\\1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 1\\2 & 2 & 3\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, r = s = t = 0$$

Eg.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$r_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + r_3 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + r_4 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -2 & 2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

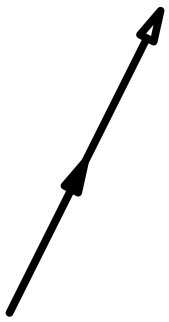
Eg. $\{X, Y\}$: *indep.* $\Rightarrow \{2X + 3Y, X - 5Y\}$: *indep.*

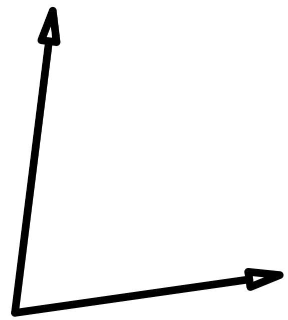
$$r(2X + 3Y) + s(X - 5Y) = O$$

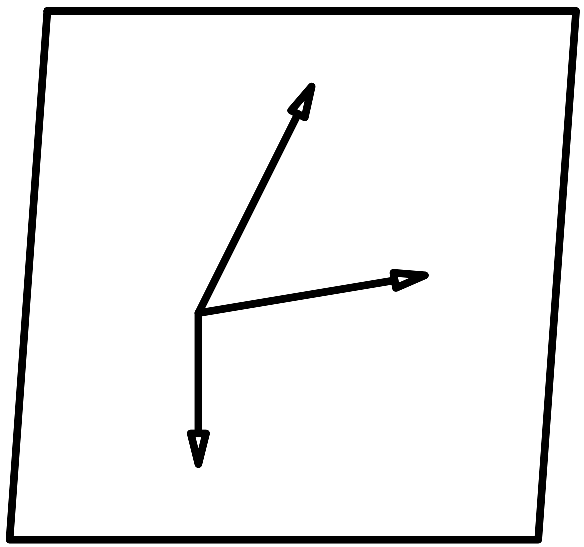
$$(2r + s)X + (3r - 5s)Y = O$$

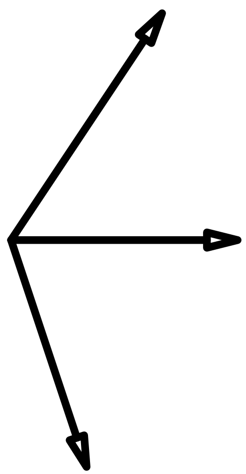
$$2r + s = 0,\ 3r - 5s = 0 \qquad r = s = 0$$

Eg.









Thm. *TFAE*

1. A *is invertible.*
2. *The columns of* A *are linearly independent.*
3. *The columns of* A *span* $\mathbf{F}^n$.
4. *The rows of* A *are linearly independent.*
5. *The rows of* A *span* $\mathbf{F}^n$.
6. $imA = \mathbf{F}^n$.
7. $kerA = O$.

Proof.

$$AX = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1C_1 + x_2C_2 + \cdots + x_nC_n$$

$$AX = O \Leftrightarrow x_1C_1 + x_2C_2 + \cdots + x_nC_n = O$$

2 $\Leftrightarrow$ $AX = O$ has only the trivial solution. $\Leftrightarrow$ $ker A = O$

$$AX = B \Leftrightarrow x_1C_1 + x_2C_2 + \cdots + x_nC_n = B$$

3 $\Leftrightarrow$ $AX = B$ has a solution for every $B \in \mathbf{F}^n$
$\Leftrightarrow$ $im A = \mathbf{F}^n$

A is invertible $\Leftrightarrow$ A^T is invertible $\square$

Eg.

$$\begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & -3 & 8 & 3 \\ -2 & 7 & -1 & 0 \\ 4 & 0 & 6 & 2 \end{bmatrix}, \quad \det A = 36 \neq 0$$

- Dimension

Def. $U \in \mathbf{F}^n$: *a subspace*
A set $\{X_1, X_2, \cdots, X_k\}$ *is a* basis *of* U, *if*

1. $\{X_1, X_2, \cdots, X_k\}$ *is linearly independent,*

2. $U = span\{X_1, X_2, \cdots, X_k\}$.

Thm. *If* $\{X_1, X_2, \cdots, X_k\}$ *and* $\{Y_1, Y_2, \cdots, Y_m\}$ *are two bases of* U, *then* $k = m$.

Def.

the number of vectors in a basis of U
$=$ *the* dimension *of* U $= \dim U$

Eg. *For* $\mathbf{F}^n$,

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, E_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix};$$

the standard basis of $\mathbf{F}^n$

Eg. *If* $\{X_1, X_2, \cdots, X_n\}$ *is a basis of* $\mathbf{F}^n$ *and* A *is invertible, then* $\{AX_1, AX_2, \cdots, AX_n\}$ *is also a basis of* $\mathbf{F}^n$.

Eg. *Consider* $AX = O$. *Recall that* $ker A$*=the span of the basic solutions. In fact, the basic solutions are linearly independent. The basic solutions form a basis for* $ker A$.

Eg. *Subspaces of* $\mathbf{R}^3$.

1. *If* $\dim U = 3$, *then* $U = \mathbf{R}^3$.

2. *If* $\dim U = 2$, *then* U *is a plane through* O.

3. *If* $\dim U = 1$, *then* U *is a line through* O.

4. *If* $\dim U = 0$, *then* $U = \{O\}$.

Thm. *Assume that* $\dim U = m = |B|$. *Then*

B *is linearly independent* $\Leftrightarrow$ B *spans* U;

in either case, B *is a basis of* U.