2.3 Matrix Inverses

- Definition

Def. A : *a square matrix*

$$B \text{ is the inverse of } A \text{ if } AB = BA = I,$$

In this case, A : invertible *or* nonsingular.

Thm 1. *If* B *and* C : *inverses of* A, *then* $B = C$.

Proof.

$$CA = I = AB.$$

$$B = IB = CAB = CI = C.$$

□

Eg 2.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $BA = I$.
Therefore, $B = A^{-1}$ *and* $A = B^{-1}$.

Eg 3. *If* $A^3 = I$, *then* $A^{-1} =$*??*

$A^3 = A^2A = AA^2 = I$. *Thus,* $A^{-1} = A^2$ *and* $(A^2)^{-1} = A$.

Eg 4. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $ad - bc \neq 0$.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- Inverses and linear systems

$$AX = B$$

If A^{-1} exists,

$$A^{-1}AX = A^{-1}B, \quad IX = A^{-1}B, \quad X = A^{-1}B.$$

A unique solution!

Thm 5. $AX = B$ *(A : square)*
If A is invertible, then it has the unique solution $X = A^{-1}B$.

Eg 6.

$$\begin{cases} x + 2y = 8 \\ 3x + 4y = 6 \end{cases}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \end{bmatrix}$$

Rmk 7.

1. *In general, finding A^{-1} is much more complicated than solving $AX = B$.*
2. *Not every square matrix has the inverse.*

- Properties

Thm 8. A, B : *square matrices*

1. $(A^{-1})^{-1} = A$
2. *If A and B are invertible,* $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1}A_1^{-1}$
4. $(A^k)^{-1} = (A^{-1})^k$
5. $(aA)^{-1} = \frac{1}{a}A^{-1}$
6. $(A^T)^{-1} = (A^{-1})^T$

Proof.

$$ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

Similarly, $B^{-1}A^{-1}AB = I$. So $(AB)^{-1} = B^{-1}A^{-1}$.

$$(A_1A_2A_3)^{-1} = [(A_1A_2)A_3]^{-1} = A_3^{-1}(A_1A_2)^{-1} = A_3^{-1}A_2^{-1}A_1^{-1}.$$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

□

A and B are invertible $\Rightarrow$ AB is invertible. Actually, the converse is also true.

A : invertible $\Leftrightarrow$ A^T : invertible

2.4 Elementary matrices

- Definition

Elementary row operations:

Type I : Interchange two rows

Type II : Multiply a row by a nonzero number

Type III : Add a multiple of a row to a different row

Def. *An* elementary matrix *is a matrix obtained from the identity matrix by an elementary row operation.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{\text{I}}{\Longrightarrow} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{\text{II}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{\text{III}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} e & f & g & h \\ a & b & c & d \\ i & j & k & l \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 5i & 5j & 5k & 5l \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} a+3i & b+3j & c+3k & d+3l \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

Thm 9. *A : $m \times n$, E : $m \times m$ elementary matrix obtained by performing some elementary row operation on I. If the same operation is performed on A, the resulting matrix is EA.*

- Inverse operation

Operation	Inverse operation
Interchange rows p and q	Interchange rows p and q
Multiply row p by $c \neq 0$	Multiply row p by $\frac{1}{c}$
Add k times row p to row q	Add $-k$ times row p to row q

$I \rightsquigarrow E_1$ by an operation ρ, $\quad I \rightsquigarrow E_2$ by the inverse operation μ

$$E_2E_1 = I \text{ and } E_1E_2 = I.$$

Thm 10. *Every elementary matrix E is invertible.*

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that
if $A \rightsquigarrow R$ in reduced row echelon form, we have

$$E_k \cdots E_2 E_1 A = R.$$

If A is a square matrix, then either $R = I$ or R has a row of zeros.

2.5 Theorem and algorithm

Thm 11. *TFAE*

1. A *is invertible.*
2. $AX = O$ *has only the trivial solution.*
3. A *can be carried to* I *by elementary row operations.*
4. A *has rank* n*.*
5. $AX = B$ *has a unique solution for every* B*.*
6. $\exists\ C$ *s.t.* $AC = I$*.*
7. A *is a product of elementary matrices.*

Proof.

(1) $\Rightarrow$ (2) : $A^{-1}AX = O$ and $X = O$.

(2) $\Rightarrow$ (3) : $A \rightsquigarrow R$ in r.r.e.f. $AX = O$ and $RX = O$ are equivalent. If $R \neq I$ has a row of zeros, $RX = O$ has infinitely many solutions, so does $AX = O$, a contradiction! Hence $R = I$.

(3) $\Rightarrow$ (4) : By definition.

(4) $\Rightarrow$ (5) : Note that the number of parameters is 0.

(5) $\Rightarrow$ (6) :

$$AX_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, AX_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, AX_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$$A \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I.$$

$$C = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}.$$

(6) $\Rightarrow$ (7) : $A \rightsquigarrow R$ in r.r.e.f. Then

$$E_k \cdots E_1 A = R.$$

If $R = I$, then $A = E_1^{-1} \cdots E_k^{-1}$. Otherwise, R has a row of zeros. In that case,

$$E_k \cdots E_1 = E_k \cdots E_1 AC = RC$$

$$RCE_1^{-1} \cdots E_k^{-1} = I$$

The left side has a row of zeros, a contradiction!

(7) $\Rightarrow$ (1) : If $A = E_1 \cdots E_k$ then $A^{-1} = E_k^{-1} \cdots E_1^{-1}$. $\square$

- Inversion method

  Assume that $AA^{-1} = I$. We write $A^{-1} = \left[X_1 X_2 \cdots X_n \right]$ where X_i : ith column of A^{-1}.

$$AA^{-1} = I \iff AX_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, AX_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, AX_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

augmented matrices : $\begin{bmatrix} & 1 \\ & 0 \\ A & 0 \\ & \vdots \\ & 0 \end{bmatrix}, \begin{bmatrix} & 0 \\ & 1 \\ A & 0 \\ & \vdots \\ & 0 \end{bmatrix}, \cdots, \begin{bmatrix} & 0 \\ & 0 \\ A & \vdots \\ & 0 \\ & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} I & X_1 \end{bmatrix}, \begin{bmatrix} I & X_2 \end{bmatrix}, \cdots, \begin{bmatrix} I & X_n \end{bmatrix}$$

using the same series of elementary row operations for each.

Simultaneously,

$$\begin{bmatrix} A & I \end{bmatrix} \Rightarrow \begin{bmatrix} I & X_1 X_2 \cdots X_n \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

Note that the last matrix is in r.r.e.f.

– inversion algorithm

$$[A \quad I] \Rightarrow [I \quad A^{-1}]$$

using elementary row operations.

If A is not invertible, then A cannot be carried to I.

Eg 12.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

Eg 13.

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} B & I \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{bmatrix}$$

Hence, B is not invertible, i.e. B is singular.