2. Matrix Algebra

2.1 Matrix Addition, Scalar Multiplication, and Transposition

- Matrix

Def. matrix *: a rectangular array of numbers*

Our numbers $\in \mathbf{R}$ or $\mathbf{C}$ Notation : $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$

– entries, rows, columns

– $m \times n$ matrix, the (i, j)-entry

Eg.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1+i & 0 & 2i \\ 4 & -2-i & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$$

Denote the (i,j)-entry by a_{ij}, and we have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Simply, we write $A = [a_{ij}]$.

– square matrix
– the main diagonal of a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

– equality

If $A = [a_{ij}]$, $B = [b_{ij}]$, then $A = B$ means $a_{ij} = b_{ij}$ for all i and j.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Leftrightarrow a = 1, b = 2, c = 3, d = 4.$$

- Matrix addition

  If $A = [a_{ij}]$, $B = [b_{ij}]$, we define

$$A + B = [a_{ij} + b_{ij}].$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \text{ Nonsense !}$$

– zero matrix O :

$$O + A = A \text{ for all } A$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

– the negative $-A$:

$$-A = [-a_{ij}] \text{ and } A + (-A) = O$$

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}, \quad -A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}$$

– difference $A - B = A + (-B) = [a_{ij} - b_{ij}]$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

- Scalar multiplication

  If $k \in \mathbf{F}$ and A is a matrix, we define

$$kA = [ka_{ij}].$$

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 2i & -1+2i & 3 \end{bmatrix}, \qquad \frac{1}{2}A = \begin{bmatrix} \frac{3}{2} & 0 & 2 \\ i & -\frac{1}{2}+i & \frac{3}{2} \end{bmatrix}$$

Thm. A, B, C : *matrices of the same size,* $k, p \in \mathbf{F}$

1. $A + B = B + A$,
2. $A + (B + C) = (A + B) + C$
3. $\exists$ O *s.t.* $O + A = A$ *for each* A.
4. *For each* A, $\exists$ $-A$ *s.t.* $A + (-A) = O$.
5. $k(A + B) = kA + kB$, $(k + p)A = kA + pA$
6. $(kp)A = k(pA)$,
7. $1 \cdot A = A$

Roughly, we can manipulate matrices as if they are variables w.r.t. addition and scalar multiplication.

Eg.

$$2(2A-3B)-3(A-B)=4A-6B-3A+3B=A-3B$$

Eg. *Find X and Y s.t.*

$$X+2Y=\begin{bmatrix}1 & 3 & -2\end{bmatrix}, \qquad X+Y=\begin{bmatrix}2 & 0 & 1\end{bmatrix}.$$

$$\Rightarrow \quad X=\begin{bmatrix}3 & -3 & 4\end{bmatrix}, \quad Y=\begin{bmatrix}-1 & 3 & -3\end{bmatrix}.$$

- Transpose of $A = [a_{ij}]$:

  $A^T = [a_{ji}]$ interchanging the rows and columns of A

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ -2 & -1 \end{bmatrix}$$

Thm.

1. $(A^T)^T = A,\qquad (kA)^T = kA^T$
2. $(A+B)^T = A^T + B^T$

Def.

$$A \ : \ \text{symmetric} \Leftrightarrow A^T = A$$

$$A \ : \ \text{skew symmetric} \Leftrightarrow A^T = -A$$

Eg.

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 4 \\ 0 & 4 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

2.2 Matrix Multiplication

- Definition

  If $A : m \times n$ and $B : n \times p$, we define the product AB to be the $m \times p$ matrix whose (i, j)-entry is computed as follows:

1. ith row of A, jth column of B
2. products of corresponding entries
3. adding the results

Eg.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 5 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \textit{Nonsense!}$$

Assume that $A : m \times n$ and $B : n' \times p$.
Only if $n = n'$, AB is defined and of the size $m \times p$.

Eg.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

– identity matrix I :

$$AI = IA = A \text{ for all } A$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cdots$$

– summation notation

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j$$

1.

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

2.

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

3.

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)$$

– product revisited

Let $A = [a_{ij}] : m \times n$, $B = [b_{ij}] : n \times p$, $AB = [c_{ij}]$.

ith row of A	:	$a_{i1}\ a_{i2}\ \cdots\ a_{in}$
jth column of B	:	$b_{1j}\ b_{2j}\ \cdots\ b_{nj}$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Thm.

1. $A(BC) = (AB)C$

2. $A(B + C) = AB + AC, \; A(B - C) = AB - AC$

3. $(B + C)A = BA + CA, \; (B - C)A = BA - CA$

4. $k(AB) = (kA)B = A(kB)$

5. $(AB)^T = B^T A^T$

Proof. Let

$A = [a_{ij}]$, $B = [b_{ij}]$, $AB = [c_{ij}]$

$A^T = [a^T_{ij}]$, $B^T = [b^T_{ij}]$, $(AB)^T = [c^T_{ij}]$, and $B^T A^T = [d_{ij}]$.

Note that

$$a^T_{ij} = a_{ji}, \; b^T_{ij} = b_{ji}, \; c^T_{ij} = c_{ji}.$$

Now

$$d_{ij} = \sum_{k=1}^{n} b^T_{ik} a^T_{kj} = \sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} a_{jk} b_{ki} = c_{ji} = c^T_{ij}.$$

□

– Danger! Watch! Beware!

1. $AB \neq BA$

2. $AB = O \not\Rightarrow A = O$ or $B = O$

3. $A \neq O$ and $AB = AC \not\Rightarrow B = C$

- Linear system and matrix

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m
\end{aligned}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Then we have $AX = B$.

For $AX = B$, we call

A : coefficient matrix, B : constant matrix.

Note that $AX = O$: homogeneous system.

Eg.

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + 2x_2 - x_3 &= -1 \end{aligned}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Clearly, $AX = B$.

Thm. $AX = B$, X_p : *a particular solution*
If X_a *is any solution to* $AX = B$, *then* $\exists$ *a solution* X_h *to* $AX = O$ *s.t.*

$$X_a = X_p + X_h.$$

Proof. X_p and X_a are given. Set $X_h = X_a - X_p$.

$$AX_h = A(X_a - X_p) = AX_a - AX_p = B - B = 0.$$

Thus X_h is a solution to $AX = O$, and $X_a = X_h + X_p$. $\square$

$$\{\text{solutions to } AX = B\} = X_p + \{\text{solutions to } AX = O\}$$

– homogeneous systems

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_4 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_4 &= 0 \\ x_2 - x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{cases} x_1, x_2, x_3 : \text{ leading variables} \\ x_4 = r : \text{ parameter} \end{cases}$$

$$\begin{cases} x_1 = -r \\ x_2 = r \\ x_3 = -r \\ x_4 = r \end{cases}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} : \text{ basic solution}$$

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1, x_2, x_4 : \text{ leading variables} \\ x_3 = r, x_5 = s : \text{ parameters} \end{cases}$$

$$X = \begin{bmatrix} -2r - s \\ -2r + s \\ r \\ -2s \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$: basic solutions

Def. *An expression*

$$a_1X_1 + a_2X_2 + \cdots + a_pX_p$$

is called a linear combination *of* $X_1, X_2, \cdots, X_p$ *(*$a_i \in \mathbf{F}$*).*

Thm. $AX = O$, n *variables, rank*$A = r$

1. $n - r$ *basic solutions*

2. Every solution is a linear combination of the basic solutions.